A Quantity Adjustment Process to Find Equilibria in Economies with Price Rigidities

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Abstract: A quantity adjustment process for an economy with price rigidities is presented. From the main theorem it follows that the quantity adjustment process converges generically to a Drèze equilibrium under standard assumptions with respect to the economy. It is possible to follow the adjustment process arbitrarily close by means of a simplicial algorithm.

Zusammenfassung: Es wird ein Quantitätsanpassungsprozeß für eine Wirtschaft mit Preisstarren dargeboten. Der wichtigste Satz impliziert, daß der Quantitätsanpassungsprozeß unter Standardannahmen in bezug auf die Wirtschaft generisch nach einem Drèze-Gleichgewicht konvergiert. Es ist möglich den Quantitätsanpassungsprozeß mittels eines Simplizialalgorithmus nachzuvollziehen.

1 Introduction

The Walrasian tatonnement process does not converge to a Walrasian equilibrium for many economies, see for instance Hahn (1982). For impossibility results for the existence of such a process, see Saari and Simon (1978). Also government interventions like the imposition of minimum wages or price setting behaviour by economic agents, see Bénassy (1993), may prevent an economy from reaching a Walrasian equilibrium. In this paper it is assumed that agents trade against some given, not necessarily Walrasian, price system. Hence, agents may face restrictions on their supply and demand. An adjustment process in trading possibilities is presented that, generically, converges to a so-called Drèze equilibrium. Moreover, the process can be followed arbitrarily close by a simplicial algorithm.

2 Economies with Price Rigidities

For $k \in \mathbb{N}$, define $I_k = \{1, \ldots, k\}$, $\mathbb{R}^k_{++} = \{x \in \mathbb{R}^k \mid x_j > 0, \forall j \in I_k\}$, $Q^k = \{q \in \mathbb{R}^k_+ \mid 0 \leq q_j \leq 1,\ \forall j \in I_k\}$, 0^k as the k-dimensional vector of zeros, and 1^k as

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the k-dimensional vector of ones. An economy with price rigidities is denoted by $\mathcal{E} = ((X^i, u^i, \omega^i)_{i \in I_M}, p)$. There are M consumers, indexed $i = 1, \ldots, M$, and $N+1$ commodities, indexed $j = 1, \ldots, N+1$. Each consumer $i \in I_M$ has a consumption set X^i, a utility function $u^i : X^i \to \mathbb{R}$, and an initial endowment $\omega^i \in \mathbb{R}^{N+1}_{+}$. The price system is assumed to be completely fixed and is given by $p \in \mathbb{R}^{N+1}_{++}$. The price of commodity $N+1$ is assumed to be equal to 1, so commodity $N+1$ is a numeraire commodity. Since at the price system p the total demand will in general not be equal to the total supply, consumers might face restrictions concerning their net demand or net supply. Assuming for the sake of simplicity that all consumers will face the same restrictions, these restrictions are represented by a so-called rationing scheme $(l, L) \in -\mathbb{R}^N_+ \times \mathbb{R}^N_+$, where l denotes the rationing scheme on supply and L the rationing scheme on demand. Given a rationing scheme $(l, L) \in -\mathbb{R}^N_+ \times \mathbb{R}^N_+$, the demand of a consumer $i \in I_M$ is denoted by $\delta^i(l, L)$ and is equal to the set of consumption bundles maximizing u^i on his budget set $B^i(l, L) = \{x^i \in X^i \mid l_j \leq x^i_j - \omega^i_j \leq L_j, \ \forall j \in I_N\}$. As is usually done in the literature about economies with price rigidities, it is assumed that there is no rationing on the market of the numeraire commodity. A Drèze equilibrium is defined as follows.

Definition 2.1 *A Drèze equilibrium of the economy $\mathcal{E}$ is an element $(x^{*1}, \ldots, x^{*M}, l^*, L^*) \in \prod_{i \in I_M} X^i \times -\mathbb{R}^N_+ \times \mathbb{R}^N_+$ satisfying:*

1. *For every $i \in I_M$, $x^{*i} \in \delta^i(l^*, L^*)$.*
2. *$\sum_{i \in I_M} x^{*i} = \sum_{i \in I_M} \omega^i$.*
3. *For every $j \in I_N$, $x^{*i'}_j - \omega^{i'}_j = l^{*i'}_j$ for some $i' \in I_M$ implies $x^{*i}_j - \omega^i_j < L^{*i}_j$, $\forall i \in I_M$, and $x^{*i'}_j - \omega^{i'}_j = L^{*i'}_j$ for some $i' \in I_M$ implies $x^{*i}_j - \omega^i_j > l^{*i}_j$, $\forall i \in I_M$.*

The first two conditions are the standard utility maximizing and market clearing conditions. The third condition implies that there is no market where some consumers are rationing on their demand while others are rationed on their supply. For the remainder of this paper, the economy $\mathcal{E}$ is assumed to satisfy the following assumptions.

A1. For every $i \in I_M$, the consumption set X^i is equal to $\mathbb{R}^{N+1}_{++}$.

A2. For every $i \in I_M$, the utility function $u^i : X^i \to \mathbb{R}$ is strictly increasing, strictly quasi-concave, three times continuously differentiable, the indifference surfaces of u^i have non-zero Gaussian curvature at every $x^i \in X^i$, and the closure of the indifference surfaces in $\mathbb{R}^{N+1}$ is a subset of $\mathbb{R}^{N+1}_{++}$.

A3. For every $i \in I_M$, the initial endowment ω^i belongs to X^i.

Along the lines of Drèze (1975) it can be shown that the economy $\mathcal{E}$ indeed has at least one Drèze equilibrium. It can also be shown that $\delta^i(l, L)$, $\forall i \in I_M$, contains a unique element for every $(l, L) \in -\mathbb{R}^N_+ \times \mathbb{R}^N_+$, denoted by $d^i(l, L)$. Moreover, it is easily seen that

$$\forall j \in I_N, \ \exists \underline{l}_j \in -\mathbb{R}_+, \ \forall (l, L) \in -\mathbb{R}^N_+ \times \mathbb{R}^N_+, \ l_j < d^i_j(l, L) - \omega^i_j, \ \forall i \in I_M, \ \text{if } l_j < \underline{l}_j \ ,$$

$$\forall j \in I_N, \ \exists \overline{L}_j \in \mathbb{R}_+, \ \forall (l, L) \in -\mathbb{R}^N_+ \times \mathbb{R}^N_+, \ L_j > d^i_j(l, L) - \omega^i_j, \ \forall i \in I_M, \ \text{if } L_j > \overline{L}_j \ .$$

For instance, choose $\underline{l}_j$ such that $\underline{l}_j < -\omega^i_j$, $\forall i \in I_M$, and $\overline{L}_j$ such that $\overline{L}_j > p{\cdot}\omega^i/p_j - \omega^i_j$, $\forall i \in I_M$. Define the functions $\widehat{l} : Q^N \to -\mathbb{R}^N_+$ and $\widehat{L} : Q^N \to \mathbb{R}^N_+$ by $\widehat{l}(q) = (2\underline{l}_1 q_1, \ldots, 2\underline{l}_N q_N)$, $\forall q \in Q^N$, and $\widehat{L}(q) = (2\overline{L}_1(1-q_1), \ldots, 2\overline{L}_N(1-q_N))$, $\forall q \in Q^N$. It is easily verified that there is no supply rationing on the market of a commodity $j \in I_N$ if $q_j \geq \frac{1}{2}$, and there is no demand rationing on the market of commodity j if $q_j \leq \frac{1}{2}$. By adjusting q the rationing scheme will be adjusted in such a way that during the adjustment Condition 3 of Definition 2.1 is always satisfied. Now the problem considered in this paper is whether there exists an adjustment process on Q^N which is such that it converges to a Drèze equilibrium irrespective of the initial value in Q^N for almost every economy satisfying the Assumptions A1-A3. Define the total excess demand function $\widehat{z} : Q^N \to \mathbb{R}^{N+1}$ by $\widehat{z}(q) = \sum_{i \in I_M} d^i(\widehat{l}(q), \widehat{L}(q)) - \sum_{i \in I_M} \omega^i$, $\forall q \in Q^N$. The proof of the following result is standard.

Theorem 2.2 *Let the economy $\mathcal{E}$ satisfy the Assumptions A1-A3. Then the function $\widehat{z}$ is continuous, for every $q \in Q^N$, $p{\cdot}\widehat{z}(q) = 0$, $q_j = 0$ implies $\widehat{z}_j(q) \geq 0$, and $q_j = 1$ implies $\widehat{z}_j(q) \leq 0$.*

For every $q^* \in Q^N$ it holds that $\widehat{z}(q^*) = 0^{N+1}$ if and only if q^* induces a Drèze equilibrium $(d^1(\widehat{l}(q^*), \widehat{L}(q^*)), \ldots, d^M(\widehat{l}(q^*), \widehat{L}(q^*)), \widehat{l}(q^*), \widehat{L}(q^*))$.

3 The Quantity Adjustment Process

The initial state of the economy is denoted by $v \in Q^N$, corresponding to the rationing scheme $(\widehat{l}(v), \widehat{L}(v))$. The quantity adjustment process is such that if there is a negative (positive) total excess demand on the market of commodity $j \in I_N$, then q_j will be decreased (increased) maximally over all commodities towards zero (one). Notice that this makes sense, since an increase in q_j decreases $\widehat{l}_j(q)$, i.e., every consumer is allowed to supply more on the market of commodity j, and decreases $\widehat{L}_j(q)$, i.e., the maximal amount a consumer can demand of commodity j is decreased.

Let $\mathbb{S}^N$ denote the set of N-dimensional sign vectors, so $\mathbb{S}^N = \{s \in \mathbb{R}^N \mid s_j \in \{-1, 0, +1\},\ \forall j \in I_N\}$. For a sign vector $s \in \mathbb{S}^N$, define $I^-(s) = \{j \in I_N \mid s_j = -1\}$, $I^0(s) = \{j \in I_N \mid s_j = 0\}$, and $I^+(s) = \{j \in I_N \mid s_j = +1\}$. Let an initial state $v \in Q^N$ be given. In order to describe the quantity adjustment process it is useful to define the set of admissible sign vectors $\mathcal{S}$ by

$$\mathcal{S} = \{s \in \mathbb{S}^N \mid v_j > 0,\ \forall j \in I^-(s),\ v_j < 1,\ \forall j \in I^+(s),\ \text{and}\ \exists j' \in I_N,\ s_{j'} \neq 0\}.$$

Notice that $\mathcal{S} \neq \emptyset$. For every $s \in \mathcal{S}$, define the sets $A(s)$, $B(s)$, and $C(s)$ by

$$\begin{array}{rlll} A(s) = \{q \in Q^N \mid \exists \mu \in [0,1], & q_j = \mu v_j, & \forall j \in I^-(s), \\ & \mu v_j \leq q_j \leq 1 - \mu(1 - v_j), & \forall j \in I^0(s), \\ & q_j = 1 - \mu(1 - v_j), & \forall j \in I^+(s)\} \ , \\ B(s) = \{q \in Q^N \mid & \widehat{z}_j(q) \leq 0, & \forall j \in I^-(s), \\ & \widehat{z}_j(q) = 0, & \forall j \in I^0(s), \\ & \widehat{z}_j(q) \geq 0, & \forall j \in I^+(s)\} \ , \\ C(s) = A(s) \cap B(s) \ . \end{array}$$

Clearly, $v \in C(s^0)$ for $s^0 \in \mathcal{S}$ with $s^0_j = -1$ if $\widehat{z}_j(v) < 0$ and $s^0_j = +1$ if $\widehat{z}_j(v) > 0$. The quantity adjustment process follows a path of points in the set C defined by $C = \cup_{s \in \mathcal{S}} C(s)$ leading from v to a state q^* inducing a Drèze equilibrium.

The quantity adjustment process has an appealing economic interpretation and can be described as follows. Consider an arbitrary initial state in the interior of Q^N (the more general case of an arbitrary initial state in Q^N is similar). Evaluate the sign of the total excess demand of the non-numeraire commodities at v. It can be shown that, generically, $z_j(v) \neq 0$, $\forall j \in I_N$. The quantity adjustment process proceeds by leaving v along the ray $A(s^0)$, where $s^0 \in \mathcal{S}$ is the vector of signs of the total excess demands at v. The process continues along this ray until for one of the commodities, say, commodity $j' \in I_N$, the total excess demand becomes equal to zero. It can be shown that, generically, either this happens for the market of exactly one commodity $j' \in I_N$ at a state $\overline{q} \in Q^N$ with $0 < \overline{q}_{j'} < 1$, or the boundary of Q^N is reached at a state $q^* \in Q^N$ satisfying $\widehat{z}(q^*) = 0^{N+1}$. In the latter case a Drèze equilibrium is reached and the process is terminated. In the former case, if $N = 1$ a Drèze equilibrium is reached and the process is terminated, and if $N \geq 2$ a path in $C(s^1)$ is followed, where $s^1 \in \mathcal{S}$ is defined by $s^1_{j'} = 0$ and $s^1_j = s^0_j$, $\forall j \in I_N \setminus \{j'\}$, so the market of commodity j' is kept in equilibrium, while for every $j \in I_N \setminus \{j'\}$, q_j is kept relatively minimal (maximal) if $\widehat{z}_j(\overline{q}) < 0$ ($\widehat{z}_j(\overline{q}) > 0$). In the general case, the quantity adjustment process follows a path in $C(s^n)$ for some $n \in \mathbb{N}$. Generically, the following cases may result. Either a state $q^* \in Q^N$ is reached satisfying $\widehat{z}(q^*) = 0^{N+1}$ and the quantity adjustment process is terminated. Or the market of a commodity $j' \in I^-(s) \cup I^+(s)$ is equilibrated, in which case a path in $C(s^{n+1})$ is followed, where $s^{n+1} \in \mathcal{S}$ is defined by $s^{n+1}_{j'} = 0$ and $s^{n+1}_j = s^n_j$, $\forall j \in I_N \setminus \{j'\}$, so the market of commodity j' is kept in equilibrium. Or $q_{j'}$ for some $j' \in I^0(s^n)$ ($1 - q_{j'}$ for some $j' \in I^0(s^n)$) becomes relatively minimal, in which case a path in $C(s^{n+1})$ is followed, where $s^{n+1} \in \mathcal{S}$ is defined by $s^{n+1}_{j'} = -1$ ($s^{n+1}_{j'} = +1$) and $s^{n+1}_j = s^n_j$, $\forall j \in I_N \setminus \{j'\}$, so the market of commodity j' is no longer kept in equilibrium. In Figs. 1, 2, and 3 the sets $A(s)$, $B(s)$, and $C(s)$, $s \in \mathcal{S}$, respectively, are illustrated for an example for $N = 2$. In the example the quantity adjustment process reaches a Drèze equilibrium after generating six sign vectors. The set C consists of the quantity adjustment process, connecting v and q^{*1}, a loop containing no states of the economy inducing Drèze equilibria, and an arc with q^{*2} and q^{*3} inducing Drèze equilibria as end points.

Formally, the quantity adjustment process is now defined as the component of the set C containing the initial state v. The quantity adjustment process is said to converge if either $\widehat{z}(v) = 0^{N+1}$, or $\widehat{z}(v) \neq 0^{N+1}$ and the component of v in C is an arc having v and a state $q^* \in Q^N$ inducing a Drèze equilibrium as its boundary points. This way of describing an adjustment process is closely related to the approach taken in Smale (1976) and van der Laan and Talman (1987).

Let $((X^i, u^i)_{i \in I_M}, p)$ satisfying the Assumptions A1 and A2 be given. Define the open subset $\mathcal{W}$ of $\mathbb{R}^{M(N+1)}_{++} \times -\mathbb{R}^N_{++} \times \mathbb{R}^N_{++}$ by $\mathcal{W} = \{(\omega^1, \ldots, \omega^M, \underline{l}, \overline{L}) \in \mathbb{R}^{M(N+1)}_{++} \times -\mathbb{R}^N_{++} \times \mathbb{R}^N_{++} \mid \forall j \in I_N,\ l_j < d^i_j(l, L) - \omega^i_j,\ \forall i \in I_M, \text{ if } l_j < \underline{l}_j, \text{ and } L_j > d^i_j(l, L) - \omega^i_j,\ \forall i \in I_M, \text{ if } L_j > \overline{L}_j\}$. Notice that an element $(\omega^1, \ldots, \omega^M, \underline{l}, \overline{L})$ determines in a unique way the economy and, given the initial state $v \in Q^N$,

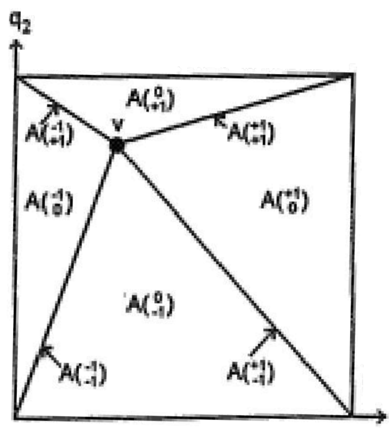

Figure 1: The sets $A(s)$, $s \in \mathcal{S}$, $N = 2$

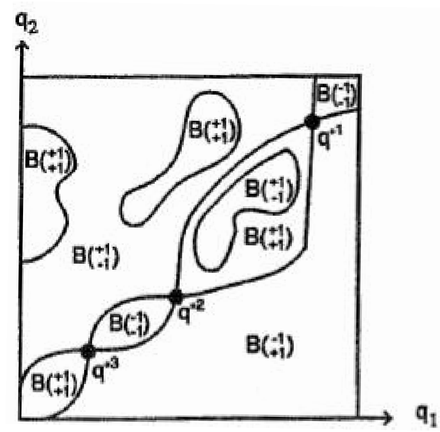

Figure 2: The sets $B(s)$, $s \in \mathcal{S}$, $N = 2$

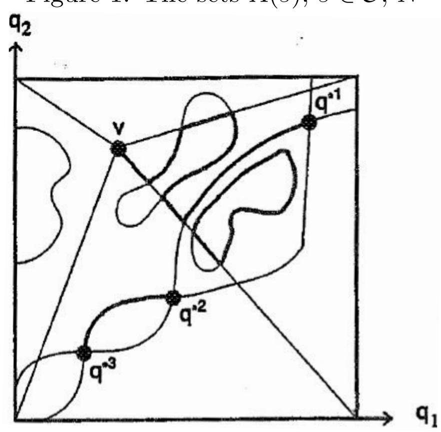

Figure 3: The sets $C(s)$, $s \in \mathcal{S}$, $N = 2$

the quantity adjustment process. Now the following result on the structure of the set C can be shown using the approach described in Herings (1994).

Theorem 3.1 *Let $((X^i, u^i)_{i \in I_M}, p)$ satisfy the Assumptions A1 and A2, and let $v \in Q^N$ be the initial state. Then the set $\mathcal{W}^* \subset \mathcal{W}$ for which the components of the set C are given by (i) the initial state v as an isolated point and inducing a Drèze equilibrium, or an arc containing v and precisely one state inducing a Drèze equilibrium as boundary points, (ii) a finite number of arcs containing precisely two states both being boundary points and inducing a Drèze equilibrium, and (iii) a finite number of loops containing neither v nor any state inducing a Drèze equilibrium, is such that $\mathcal{W} \setminus \mathcal{W}^*$ has a closure in $\mathcal{W}$ with Lebesgue measure zero.*

From Theorem 3.1 it follows that, generically, the quantity adjustment process converges to a Drèze equilibrium, i.e., for almost every $(\omega^1, \ldots, \omega^M, \underline{l}, \overline{L}) \in \mathcal{W}$ the component of v in C is an arc. It is not difficult to construct counterexamples showing that the set C is not necessarily an arc for every $(\omega^1, \ldots, \omega^M, \underline{l}, \overline{L}) \in \mathcal{W}$.

Nevertheless, the following result shows that the component of v in C always contains at least one state inducing a Drèze equilibrium.

Theorem 3.2 *Let the economy $\mathcal{E}$ satisfy the Assumptions A1-A3 and let $v \in Q^N$ be the initial state. Then the component of v in C contains a state inducing a Drèze equilibrium.*

It is possible to follow the quantity adjustment process numerically by the product-ray algorithm described in Doup and Talman (1987). In order to be able to follow the quantity adjustment process for all economies, the pivoting rules of Doup and Talman have to be modified into lexicographic pivoting rules using the approach as described in Herings et al. (1994). For every triangulation Σ of Q^N, this modified algorithm yields a piecewise linear path of points, denoted by $\gamma_\Sigma : [0,1] \to Q^N$, such that $\gamma_\Sigma(0) = v$ and $\gamma_\Sigma(1)$ is a state of the economy inducing an approximate Drèze equilibrium. Denote the component of v in C by Γ. The following result shows that the modified algorithm converges to the quantity adjustment process and can be shown along the lines of Herings et al. (1994). As is shown there, Theorem 3.3 can be used to prove Theorem 3.2.

Theorem 3.3 *Let the economy $\mathcal{E}$ satisfy the Assumptions A1-A3 and let $v \in Q^N$ be the initial state. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if the mesh size of Σ is smaller than δ, then $\forall t \in [0,1]$, $\exists q \in \Gamma$, $\|\gamma_\Sigma(t) - q\|_\infty < \varepsilon$.*

References

Bénassy, J.-P.: Nonclearing markets: microeconomic concepts and macroeconomic applications. Journal of Economic Literature **31** (1993) 732-761

Doup, T.M., Talman, A.J.J.: A new simplicial variable dimension algorithm to find equilibria on the product space of unit simplices. Mathematical Programming **37** (1987) 319-355

Drèze, J.H.: Existence of an exchange equilibrium under price rigidities. International Economic Review **16** (1975) 301-320

Hahn, F.H.: Stability. In Arrow, K.J., Intriligator, M.D. (eds.): Handbook of mathematical economics, volume II, North-Holland, Amsterdam (1982) pp. 745-793

Herings, P.J.J.: A globally and universally stable quantity adjustment process for an exchange economy with price rigidities. CentER Discussion Paper 94111, CentER, Tilburg University, Tilburg (1994) 1-37

Herings, P.J.J., Talman, A.J.J., Yang, Z.: The computation of a continuum of constrained equilibria. CentER Discussion Paper 9438, CentER, Tilburg University, Tilburg (1994) 1-25 Mathematics of Operations Research (to appear)

Laan, G. van der, Talman, A.J.J.: A convergent price adjustment process. Economics Letters **23** (1987) 119-123

Saari, D.G., Simon, C.P.: Effective price mechanisms. Econometrica **46** (1978) 1097-1125

Smale, S.: A convergent process of price adjustment and global Newton methods. Journal of Mathematical Economics **3** (1976) 107-120