

# Closed Form Solutions for a Game of Macroeconomic Policy in a Two-Party System\*

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## Zusammenfassung

Der sehr einflußreiche Beitrag von Alesina (1987) modelliert die Interaktion von politischen Kandidaten mit unterschiedlichen Zielen bezüglich Inflation und Arbeitslosigkeit als ein wiederholtes Spiel. Wir geben die analytischen Lösungen des Spiels an und vergleichen sie mit den approximativen Lösungen von Alesina.

## Abstract

The very influential paper, Alesina (1987), models the interaction of political candidates with different objectives concerning inflation and unemployment as a repeated game. We derive the exact closed form expressions related to this game and compare them with the results obtained by Alesina, who used an approximation.

## 1 Introduction

In Alesina (1987) the interaction of two political parties with different objectives concerning inflation and unemployment is considered. A repeated game to model this interaction is formulated. For the corresponding static game the efficient frontier and a threat point for both political parties are derived. It is shown that the Nash bargaining solution is a function of the probability that a political party wins the elections. However, it is difficult to derive the closed form solution for this function, which is also needed to derive the value the discount factor must have in order to be able to support the Nash bargaining solution as a subgame perfect equilibrium in the repeated game. Alesina solves these problems by using an approximation of the closed form solution. In this paper the exact closed form expressions for the Nash bargaining solution and the discount factors sustaining the Nash bargaining solution are derived, and their properties are analysed.

## 2 The Model

Consider an infinite horizon model with a countable number of periods, denoted by  $t = 0, 1, \dots$ . There are three players in the model, two political parties, denoted by  $D$  and  $R$ , and a wage-setting institution. In period  $t$  polls are taken which reveal that party  $D$  will win the elections in period  $t + 1$  with probability  $P$ , where  $0 < P < 1$ , and that party  $R$  will win the elections with probability  $1 - P$ . For simplicity,  $P$  is assumed to be independent of  $t$ . Immediately after the polls, wages for period  $t + 1$  are set by the wage-setting institution. After the elections in period  $t + 1$ , the elected party chooses the level of inflation in period  $t + 1$ . Then the polls of period  $t + 1$  take place, and so on. Let  $\Pi_t^D$  ( $\Pi_t^R$ ) denote the level of inflation party  $D$  (party  $R$ ) chooses if it would be elected in period  $t$  and let  $\Pi_t$  be the resulting level of inflation in period  $t$ .

It is assumed that the wage-setting institution attempts to set the proportional increase in the wage rate for period  $t$ ,  $w_t$ , equal to the expected inflation level in that period, given all information

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available in period  $t - 1$ . This guarantees that the expected level of real wages is kept constant. With  $q$ ,  $0 < q < 1$ , denoting some given discount rate, the cost of the wage-setting institution is given by

$$\sum_{t=0}^{\infty} q^t (w_t - P\Pi_t^D - (1 - P)\Pi_t^R)^2.$$

It is assumed that, compared to party  $R$ , for party  $D$  unemployment is an important issue, while for this party it is less important to fight against inflation. A positive relationship between the rate of output growth and the amount of inflation not expected by the wage setting institution is derived using either the Lucas supply function or the Phillips curve. The political parties face the following costs,

$$\begin{aligned} C^D &= \sum_{t=0}^{\infty} q^t \left( \frac{1}{2}\Pi_t^2 - b(\Pi_t - w_t) - c\Pi_t \right), & \text{for party D,} \\ C^R &= \sum_{t=0}^{\infty} q^t \left( \frac{1}{2}\Pi_t^2 \right), & \text{for party R,} \end{aligned} \quad (1)$$

where  $b$  and  $c$  are given parameters satisfying  $b \geq 0$ ,  $c \geq 0$ , and  $b + c > 0$ . If  $b$  is high, then political party  $D$  considers unemployment to be very important. If  $c$  is high, then the level of inflation desired by political party  $D$  is high.

In Alesina (1987) the following results are derived. In the one-shot Nash equilibrium the level of inflation and the wage rate are equal to  $\Pi_t^{*D} = b + c$ ,  $\forall t \in \mathbb{N}$ ,  $\Pi_t^{*R} = 0$ ,  $\forall t \in \mathbb{N}$ , and  $w_t^* = P(b + c)$ ,  $\forall t \in \mathbb{N}$ , respectively. The one-shot Nash equilibrium is not on the efficient frontier of the game. The part of the efficient frontier which Pareto dominates the one-shot Nash equilibrium corresponds to the points  $\Pi_t^D = \Pi_t^R = w_t = \frac{c}{1+\theta}$ ,  $\forall t \in \mathbb{N}$ , with  $\theta \geq 0$ , where, in order to guarantee individual rationality, there might exist a lower bound,  $\underline{\theta}(P)$ , and an upper bound,  $\bar{\theta}(P)$ , on  $\theta$  which both depend on  $P$ . Notice that  $\theta \geq 0$  determines a unique point of the efficient frontier. In order to investigate the possibilities for cooperation, the approach of Friedman (1971) is followed. Let some individual rational point on the efficient frontier be given. Let this point correspond with both parties playing  $\Pi'$  in all periods. In order to sustain this point as a non-cooperative equilibrium both players use the following strategies. In the first period play  $\Pi'$ . In subsequent periods play  $\Pi'$  as long as both parties have always played  $\Pi'$  in the past and play the one-shot Nash equilibrium in all other cases. It should be noticed that harsher punishment strategies might exist (see Abreu (1988)). In order to sustain a point on the efficient frontier corresponding with some given individually rational  $\theta$  as a subgame perfect equilibrium, using the strategies described above, the discount factor  $q$  has to satisfy simultaneously the following constraints,

$$\left( \frac{c}{1+\theta} \right)^2 - \frac{2bc}{1+\theta} (1 - q) - \frac{2c^2}{1+\theta} + (b + c)^2 (1 - q - Pq) + 2Pqc (b + c) \leq 0, \quad (2)$$

$$\left( \frac{c}{1+\theta} \right)^2 - Pq (b + c)^2 \leq 0. \quad (3)$$

Constraint (2) guarantees that party  $D$  has no incentive to deviate from  $\theta$ , while constraint (3) assures that party  $R$  plays cooperatively. The Nash bargaining solution is used to choose a point on the efficient frontier (see Nash (1953)). The disagreement point is given by the individual most-preferred policies, which correspond to  $\Pi_t^D = c$  and  $\Pi_t^R = 0$ . It should be remarked that the choice of the disagreement point is rather ambiguous (see for example Binmore, Rubinstein, and Wolinsky (1986)). In Alesina (1987) it is derived that the Nash bargaining solution of this game,  $\theta^*$ , satisfies

$$P = \frac{3\theta^* + 1}{(1 + \theta^*)^3}. \quad (4)$$

Alesina (1987) states “...one would need a closed form for the Nash bargaining solution ( $\theta^*$ ). Lacking this, an approximation can be considered as an example:  $\theta^*(P) \simeq \frac{1-P}{P}$ .” Then Alesina completes the analysis using this approximation.

### 3 Derivation of the Closed Form Solutions

To find a closed form expression for the Nash bargaining solution, one has to find the inverse of equation (4). Using a method described in Uspensky (1948) to solve a cubic equation, it is possible to determine this inverse. Since it is assumed that  $P > 0$  it is possible to rewrite equation (4) and obtain

$$\theta^{*3} + 3\theta^{*2} + \left(3 - \frac{3}{P}\right) \theta^* + 1 - \frac{1}{P} = 0. \quad (5)$$

So, to obtain the solution  $\theta^*$  as a function of  $P$  one has to find the zero points of the function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by  $f(\theta) = \theta^3 + 3\theta^2 + \left(3 - \frac{3}{P}\right) \theta + 1 - \frac{1}{P}$ ,  $\forall \theta \in \mathbb{R}_+$ . Substitution of  $\theta = x - 1$  in equation (5) yields

$$x^3 - \frac{3}{P}x + \frac{2}{P} = 0. \quad (6)$$

Substitute  $x = y + z$  in equation (6). This yields  $(y + z)^3 - \frac{3}{P}(y + z) + \frac{2}{P} = 0$ , or, equivalently,

$$y^3 + z^3 + \left(-\frac{3}{P} + 3yz\right)(y + z) + \frac{2}{P} = 0. \quad (7)$$

Since one degree of freedom is left with respect to the choice of  $y$  and  $z$  it is possible to choose  $y$  and  $z$  in such a way that  $yz = \frac{1}{P}$ , so  $y^3z^3 = \frac{1}{P^3}$  and equation (7) gives  $y^3 + z^3 = -\frac{2}{P}$ . The latter two expressions yield that  $y^3$  has to be a solution of the equation

$$y^6 + \frac{2}{P}y^3 + \frac{1}{P^3} = 0,$$

which is quadratic in  $y^3$ . If solutions of this equation are denoted by  $Y$  and  $Z$ , then

$$Y = -\frac{1}{P} + i\sqrt{\frac{1}{P^3} - \frac{1}{P^2}} \text{ and } Z = -\frac{1}{P} - i\sqrt{\frac{1}{P^3} - \frac{1}{P^2}}, \quad (8)$$

where it should be remembered that  $0 < P < 1$ . Using symmetry considerations it can be assumed that  $Y = y^3$  en  $Z = z^3$ . Notice that by the theory of polynomials, there are three solutions for  $y$  satisfying  $y^3 = Y$ . Let  $\sqrt[3]{Y}$  be one of these solutions. Equivalently, there are three solutions for  $z$  satisfying  $z^3 = Z$ . Let  $\sqrt[3]{Z}$  be the unique solution for  $z$  out of the three mentioned above, satisfying  $\sqrt[3]{Y} \cdot \sqrt[3]{Z} = \frac{1}{P}$ . If  $\psi = \frac{-1+i\sqrt{3}}{2}$ , then it can easily be verified that  $\psi^3 = 1$ . Hence, for the solutions  $y$  and  $z$  it has to hold that  $y = \sqrt[3]{Y}$  or  $y = \psi\sqrt[3]{Y}$  or  $y = \psi^2\sqrt[3]{Y}$ , and  $z = \sqrt[3]{Z}$  or  $z = \psi\sqrt[3]{Z}$  or  $z = \psi^2\sqrt[3]{Z}$ . Since  $y$  and  $z$  have to satisfy  $yz = \frac{1}{P}$ , not all combinations of  $y$  and  $z$  are allowed. It is easily seen that the possible solutions of equation (6) are given by  $x_1 = \sqrt[3]{Y} + \sqrt[3]{Z}$ ,  $x_2 = \psi\sqrt[3]{Y} + \psi^2\sqrt[3]{Z}$ , and  $x_3 = \psi^2\sqrt[3]{Y} + \psi\sqrt[3]{Z}$ . Equation (8) shows that for  $P$  between 0 and 1,  $Y$  and  $Z$  are complex numbers, which implies that  $\sqrt[3]{Y}$  and  $\sqrt[3]{Z}$  are complex numbers. An arbitrary complex number  $\alpha + \beta i$ , where  $\alpha$  and  $\beta$  are reals, can be rewritten as  $r(\cos\varphi + i\sin\varphi)$  where  $r = \sqrt{\alpha^2 + \beta^2}$  and  $\cos\varphi = \frac{\alpha}{r}$ . For  $Y$  it holds that  $\|Y\|_2 = \sqrt{\left(-\frac{1}{P}\right)^2 + \left(\frac{1}{P^3} - \frac{1}{P^2}\right)} = \frac{1}{P\sqrt{P}}$ . So,  $Y = \frac{1}{P\sqrt{P}}(\cos\varphi + i\sin\varphi)$ , where

$$\varphi = \arccos\left(P\sqrt{P} \cdot \frac{-1}{P}\right) = \arccos\left(-\sqrt{P}\right). \quad (9)$$

Using the formula of De Moivre a possible solution for  $\sqrt[3]{Y}$  is given by  $\sqrt[3]{Y} = \frac{1}{\sqrt{P}}\left(\cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}\right)$ , and then  $\sqrt[3]{Z} = \frac{1}{\sqrt{P}}\left(\cos\frac{\varphi}{3} - i\sin\frac{\varphi}{3}\right)$ . Moreover, it holds that  $\psi = -\frac{1}{2} + i\frac{1}{2}\sqrt{3} = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$ . Consequently, the solutions of equation (6) are given by

$$\begin{aligned} x_1 &= \sqrt[3]{Y} + \sqrt[3]{Z} = \frac{1}{\sqrt{P}}\left(\cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}\right) + \frac{1}{\sqrt{P}}\left(\cos\frac{\varphi}{3} - i\sin\frac{\varphi}{3}\right) \\ &= \frac{2}{\sqrt{P}}\cos\frac{\varphi}{3}, \end{aligned}$$

$$\begin{aligned} x_2 &= \psi\sqrt[3]{Y} + \psi^2\sqrt[3]{Z} = \left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)\left(\cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}\right)\frac{1}{\sqrt{P}} + \left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)^2\left(\cos\frac{\varphi}{3} - i\sin\frac{\varphi}{3}\right)\frac{1}{\sqrt{P}} \\ &= \frac{2}{\sqrt{P}}\cos\left(\frac{2}{3}\pi + \frac{\varphi}{3}\right), \end{aligned}$$

$$\begin{aligned} x_3 &= \psi^2\sqrt[3]{Y} + \psi\sqrt[3]{Z} = \left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)^2\left(\cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}\right)\frac{1}{\sqrt{P}} + \left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)\left(\cos\frac{\varphi}{3} - i\sin\frac{\varphi}{3}\right)\frac{1}{\sqrt{P}} \\ &= -\frac{2}{\sqrt{P}}\cos\left(\frac{1}{3}\pi + \frac{\varphi}{3}\right). \end{aligned}$$

For  $P$  between 0 and 1 the three solutions are real numbers. Substitution of equation (9) and taking into account that  $\theta = x - 1$  yields the solutions of equation (5),

$$\begin{aligned}\theta_1^* &= \frac{2}{\sqrt{P}} \cos \left( \frac{\arccos(-\sqrt{P})}{3} \right) - 1, \\ \theta_2^* &= \frac{2}{\sqrt{P}} \cos \left( \frac{2}{3}\pi + \frac{\arccos(-\sqrt{P})}{3} \right) - 1, \\ \theta_3^* &= -\frac{2}{\sqrt{P}} \cos \left( \frac{1}{3}\pi + \frac{\arccos(-\sqrt{P})}{3} \right) - 1.\end{aligned}$$

If  $0 < P < 1$ , then  $\theta_1^* > 0$ ,  $\theta_2^* < -3$ , and  $-\frac{1}{3} < \theta_3^* < 0$ . Since  $\theta^*$  has to be positive, the closed form for the Nash bargaining solution is given by

$$\theta^*(P) = \frac{2}{\sqrt{P}} \cos \left( \frac{\arccos(-\sqrt{P})}{3} \right) - 1, \quad \forall P \in (0, 1). \quad (10)$$

If one is interested in the level of the discount factor needed to sustain the Nash bargaining solution, then one has to substitute the result of equation (10) into equations (2) and (3). Define the function  $g : (0, 1) \rightarrow \mathbb{R}$  by

$$g(P) = \theta^*(P) + 1 = \frac{2}{\sqrt{P}} \cos \left( \frac{\arccos(-\sqrt{P})}{3} \right), \quad \forall P \in (0, 1).$$

Then one obtains the following constraints that have to be satisfied both:

$$q \geq \frac{(c - (b + c)g(P))^2}{((b + c)^2 + P(b^2 - c^2))(g(P))^2 - 2bcg(P)}, \quad (11)$$

$$q \geq \frac{c^2}{(b + c)^2 (g(P))^2 P}. \quad (12)$$

If the constraint in equation (11) (equation (12)) is satisfied, then party  $D$  (party  $R$ ) has no incentive to deviate.

## 4 Properties of the Closed Form Solutions

Suppose that the probability that party  $D$  wins the elections,  $P$ , is given. Let  $\theta^*(P)$  denote the Nash bargaining solution and  $\tilde{\theta}^*(P)$  the Nash bargaining solution using the approximation of Alesina. The minimum value of  $q$  which satisfies (2) when  $\theta^*(P)$  ( $\tilde{\theta}^*(P)$ ) is substituted will be denoted by  $q^{*D}(P)$  ( $\tilde{q}^{*D}(P)$ ), and the minimum value of  $q$  which satisfies (3) after substituting  $\theta^*(P)$  ( $\tilde{\theta}^*(P)$ ) is denoted by  $q^{*R}(P)$  ( $\tilde{q}^{*R}(P)$ ). In Figure 1 a comparison is made between  $\theta^*(P)$  and  $\tilde{\theta}^*(P)$ . The closed form solution is given by the solid line and the approximation by the broken line. If  $0 < P < \frac{1}{2}$ , then  $\tilde{\theta}^*(P) > \theta^*(P)$ . This means that if the probability that party  $D$  wins the elections is less than one half, then the weight given to party  $R$  in the Nash bargaining solution is overestimated by the approximation. On the other hand, if  $\frac{1}{2} < P < 1$ , then  $\tilde{\theta}^*(P) < \theta^*(P)$ . So  $\theta^*(P)$  is always closer to 1 than  $\tilde{\theta}^*(P)$  and therefore the Nash bargaining solution is less sensitive to  $P$  than suggested by the approximation. The approximation is bad if  $P$  is small. It is easily shown that it is of the wrong order if  $P \downarrow 0$ . It holds that  $\tilde{\theta}^*(P) = O(P^{-1})$  if  $P \downarrow 0$  while  $\theta^*(P) = O(P^{-\frac{1}{2}})$  if  $P \downarrow 0$ .

In Figure 2a a picture is drawn in order to compare the effect of the approximation on the discount factors for which the Nash bargaining solution is sustainable and corresponds with Figure II in Alesina (1987). The constraints the discount factors have to satisfy are drawn. The constraints induced by the closed form expression are given by the solid lines and the constraints induced by the approximation are given by the broken lines. The values of  $q$  which sustain the cooperative solution are the ones in the area above the solid lines. The case where  $b = 0$  and  $c$  is an arbitrary positive

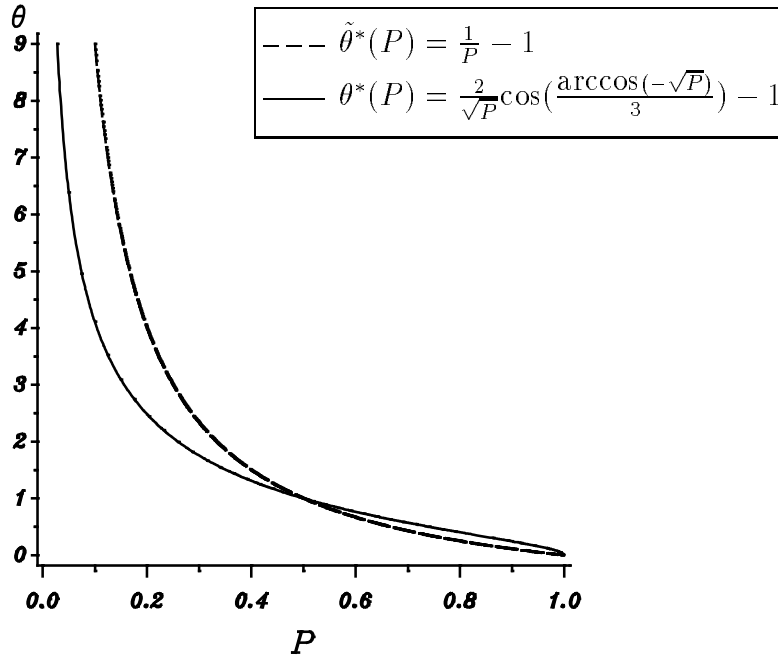


Figure 1. Nash bargaining solution.

number is considered first. The decreasing lines correspond with constraint (11) and the increasing lines correspond with constraint (12). Clearly, the differences between the solid lines and the broken lines are considerable. If  $P \downarrow 0$ , then  $\tilde{q}^{*R}(P) \rightarrow 0$ , so according to the approximation party  $R$  has no incentive to deviate. However,

$$\lim_{P \downarrow 0} q^{*R}(P) = \lim_{P \downarrow 0} \frac{1}{(g(P))^2 P} = \frac{1}{4(\cos(\frac{\arccos(0)}{3}))^2} = \frac{1}{3}. \quad (13)$$

So  $q > \frac{1}{3}$  guarantees that party  $R$  will not deviate in case  $P$  is close to 0. The case where  $P \uparrow 1$  is predicted correctly by the approximation with respect to party  $R$ . If  $P \uparrow 1$ , then the values of  $q$  for which party  $D$  deviates are considerably different from the ones suggested by the approximation. According to the approximation  $q > 0$  guarantees that party  $D$  does not deviate in case  $P$  is close to 1, while this is only true if  $q > \frac{1}{3}$ . The area of values of  $q$  which sustain the cooperative solution is larger than the area suggested by the approximation. This means that the approximation underestimates (for every value of  $P$ ) the possibilities of cooperation between the parties. The result that cooperation is easiest sustained if  $P = \frac{1}{2}$  remains true however.

In Figure 2b the case where  $b = 1$  and  $c = 5$  is considered. This figure corresponds with Figure III in Alesina (1987). Again, the constraints induced by the closed form expression are given by the solid lines and the constraints induced by the approximation are given by the broken lines. Notice that the difference between the exact solution and the approximation is considerable. Interestingly, the minimum value of  $q$  which guarantees that party  $D$  does not deviate is no longer decreasing as a function of  $P$ . There is a local minimum of this function at about 0.83. The fact that for party  $D$  the minimum value of  $q$  is not necessarily decreasing as a function of  $P$  is not very surprising. If  $P$  increases, then  $\theta^*(P)$  decreases, which means that party  $D$  obtains a better point on the efficient frontier. This makes party  $D$  less willing to deviate. However, on the other hand an augmentation in the probability that political party  $D$  wins the elections increases the pay-offs of party  $D$  in the one-shot Nash solution in case  $c > b$ . This makes party  $D$  more willing to deviate. Hence, the effects of an increase of  $P$  are not clear. The minimum value of  $q$  which makes party  $R$  to cooperate is increasing as a function of  $P$ , irrespective the values of  $b$  and  $c$ . This follows immediately from the observation that the right-hand side of (12) is increasing in  $P$ .

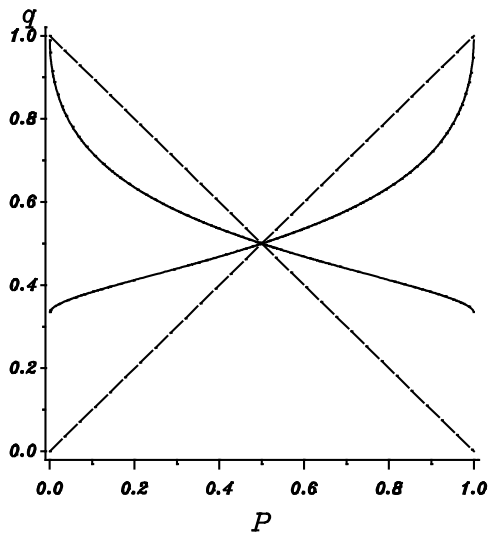


Figure 2a. Discount factors sustaining Nash bargaining solution,  $b = 0$ ,  $c > 0$ .

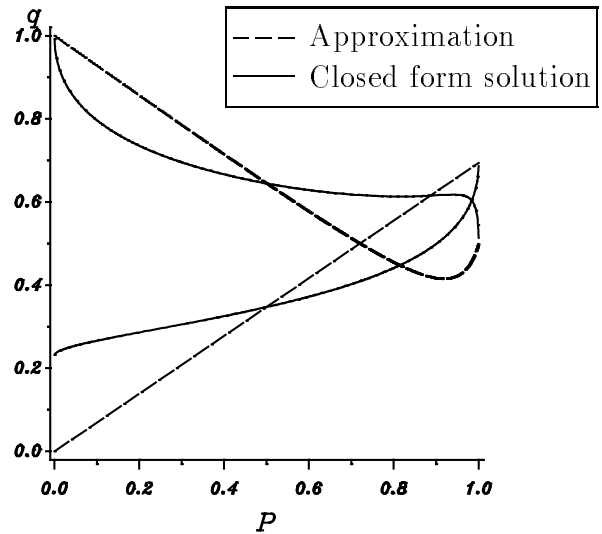


Figure 2b. Discount factors sustaining Nash bargaining solution,  $b = 1$ ,  $c = 5$ .

Finally, it should be remarked that in both Figures 2a and 2b with respect to party  $D$ , the minimum value of  $q$  is overestimated by the approximation for  $0 < P < \frac{1}{2}$  and the minimum value of  $q$  is underestimated for  $\frac{1}{2} < P < 1$ . With respect to party  $R$  the opposite holds. It will be shown that this is the case for all values of  $b$  and  $c$  satisfying  $b \geq 0$ ,  $c \geq 0$ , and  $b + c > 0$ . Using that the approximation overestimates  $\theta^*(P)$  if  $0 < P < \frac{1}{2}$  and underestimates  $\theta^*(P)$  if  $\frac{1}{2} < P < 1$ , the results follow if it can be shown that the expression in the right-hand side of (11),  $\frac{(c-(b+c)\tilde{g})^2}{((b+c)^2+P(b^2-c^2))(\tilde{g})^2-2bc\tilde{g}}$ , is increasing in  $\tilde{g}$  for  $\tilde{g} > 1$  and the expression in the right-hand side of (12),  $\frac{c^2}{(b+c)^2(\tilde{g})^2P}$ , is decreasing in  $\tilde{g}$  for  $\tilde{g} > 1$ . The latter is immediate. The former is easily shown by computing the derivative with respect to  $\tilde{g}$  and showing it is non-negative.

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