

## Two simple proofs of the feasibility of the linear tracing procedure<sup>★</sup>

P. Jean-Jacques Herings

Department of Econometrics and CentER, Tilburg University, P.O. Box 90153,  
NL-5000 LE Tilburg, THE NETHERLANDS (e-mail: Herings@kub.nl)

Received: June 8, 1998; revised version: November 8, 1998

**Summary.** Theories of equilibrium selection in non-cooperative games, as well as the notion of risk dominance, depend heavily on the so-called linear tracing procedure. This is the first paper to give direct, simple proofs of the feasibility of the linear tracing procedure. The first proof utilizes a result that is related to Kakutani's fixed point theorem and that is an extension of Browder's fixed point theorem. The second proof shows that it is even possible to avoid the use of correspondences.

**Keywords and Phrases:** Non-cooperative game theory, Tracing procedure, Equilibrium selection, Browder's fixed point theorem.

**JEL Classification Number:** C72.

### 1 Introduction

The existence of multiple Nash equilibria is a common feature of many non-cooperative  $N$ -person games. Sometimes it is possible to reduce this multiplicity of equilibria using some refinement of the concept of Nash equilibrium. However, even when using some refinement of the Nash equilibrium concept, there exists a large class of games having many equilibria that cannot be ruled out.

The equilibrium selection theory as presented in Harsanyi and Selten (1988) has the nice feature that it selects a unique Nash equilibrium for every game. To a large extent the Harsanyi-Selten theory is based on the linear tracing procedure, a mathematical procedure introduced in Harsanyi (1975) that yields a unique Nash

---

<sup>★</sup> I would like to thank Eric van Damme for helpful comments and suggestions. The research of P.J.J. Herings has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

equilibrium for almost every game. To select also a unique equilibrium for the remaining measure zero set of games, Harsanyi and Selten use a construct called the logarithmic tracing procedure. The linear tracing procedure is used repeatedly in their equilibrium selection theory to find a unique solution of so-called basic games and to define risk-dominance relationships between Nash equilibria. These risk dominance relationships play also an important role in evolutionary game theory. The linear tracing procedure is also used in other theories of equilibrium selection that are usually closely related to the ideas of Harsanyi and Selten (1988) and that are applied to several interesting economic models, see for instance van Damme (1996), and van Damme and Hurkens (1997).

The linear tracing procedure models a process of convergent expectations by which rational players will come to adopt and expect each other to adopt on a particular Nash equilibrium as a solution for a given game. Before applying the tracing procedure, players are assumed to have a common probability distribution expressing their expectations about the strategy choices of the other players. This common probability distribution is called a prior. In the naive Bayesian approach all players would choose best replies to the prior and would in this way reach a strategy combination that does not constitute a Nash equilibrium in general. In the linear tracing procedure the information on the best replies is only gradually fed back into the expectations of the players. As the linear tracing procedure proceeds, both the prior and the best responses will gradually change until both converge to some Nash equilibrium of the game.

The linear tracing procedure is formally defined as the solution to a system of equations being a homotopy. If the set of solutions to this homotopy connects the prior to at least one Nash equilibrium, then the linear tracing procedure is said to be feasible. The only proof of feasibility of the linear tracing procedure is given in Schanuel, Simon, and Zame (1991). To show the feasibility of the linear tracing procedure, they show first in an ingenious way that the logarithmic tracing procedure connects the prior beliefs to exactly one Nash equilibrium.<sup>1</sup> From this result, the feasibility of the linear tracing procedure follows by an easy limit argument. The proofs related to the logarithmic tracing procedure are very long and use heavy mathematical machinery from the field of algebraic geometry.

We show that if one is only interested in the feasibility of the linear tracing procedure, then there is no need to invoke the logarithmic tracing procedure. Indeed, we will give two very short proofs of the feasibility of the linear tracing procedure that do not involve the logarithmic tracing procedure and that are based on theorems related to Brouwer's fixed point theorem and Kakutani's fixed point theorem, known as Browder's fixed point theorem and an extension of it given by Mas-Colell (1974). The second proof combines the idea of the first proof with the satisficing principle as introduced in Geanakoplos (1996). In this way it is even possible to avoid working with correspondences. These short proofs lead to more insight into the functioning of the linear tracing procedure and may be helpful for computational purposes.

<sup>1</sup> This is the result aimed at by Schanuel, Simon, and Zame (1991). The feasibility of the linear tracing procedure can be seen as a by-product of this result.

### 2 Preliminaries

For  $m \in \mathbb{N}$ , let  $I_m$  denote the set of integers  $\{1, \dots, m\}$ , let  $\mathbb{R}_+^m$  be the non-negative orthant of the  $m$ -dimensional Euclidean space,  $\mathbb{R}_+^m = \{x \in \mathbb{R}^m \mid x_i \geq 0, \forall i \in I_m\}$ , and let  $\Delta^m$  denote the  $(m - 1)$ -dimensional unit simplex,  $\Delta^m = \{x \in \mathbb{R}_+^m \mid \sum_{i \in I_m} x_i = 1\}$ .

Consider a non-cooperative  $N$ -person normal form game  $\Gamma = (\Phi_1, \dots, \Phi_N, R_1, \dots, R_N)$ . Each player  $i \in I_N$  has  $M_i$  pure strategies that are numbered. The  $k$ -th pure strategy of player  $i$  is denoted by  $(i, k)$ . The set of pure strategies of player  $i$  is denoted by  $\Phi_i$ . The total number of strategies is given by  $M = \sum_{i \in I_N} M_i$ . The set of all pure strategy combinations is given by  $\Phi = \prod_{i \in I_N} \Phi_i$ . The function  $R_i : \Phi \rightarrow \mathbb{R}$  denotes the payoff function of a player  $i$  and it is extended in the standard way to the set of all mixed strategy combinations  $\Delta = \prod_{i \in I_N} \Delta^{M_i}$ . Here we identify all probability distributions on  $\Phi_i$  with  $\Delta^{M_i}$ , where for  $s_i \in \Delta^{M_i}$  the probability assigned to pure strategy  $(i, k)$  is given by  $s_{ik}$ . Given a mixed strategy combination  $s \in \Delta$  and a mixed strategy  $\bar{s}_i \in \Delta_i$ , we denote by  $s \setminus \bar{s}_i$  the mixed strategy combination that results from replacing  $s_i$  by  $\bar{s}_i$ .

For the remainder of the paper, some game  $\Gamma$  is assumed to be given. A mixed strategy combination  $s \in \Delta$  is said to be a Nash equilibrium of  $\Gamma$  if it is a best reply against itself. The set of Nash equilibria of a game  $\Gamma$  is denoted by  $NE(\Gamma)$ . Moreover, a probability distribution  $p \in \Delta$ , called the prior, is given for the remainder of this paper. The prior describes the initial beliefs of all players about the strategies played by the other players. The prior is assumed to be the same for all players, and the determination of the prior is part of the equilibrium selection theory of Harsanyi and Selten (1988). For every  $t \in [0, 1]$ , the linear tracing procedure generates a Nash equilibrium of the game  $\Gamma^t = (\Phi_1, \dots, \Phi_N, H_1^t, \dots, H_N^t)$ , where the payoff function  $H_i^t : \Delta \rightarrow \mathbb{R}$  of player  $i$  is defined by

$$H_i^t(s) = tR_i(s) + (1 - t)R_i(p \setminus s_i).$$

The game  $\Gamma^0$  corresponds to a trivial game, where all players believe that all their opponents play with probability 1 according to the prior beliefs. The game  $\Gamma^1$  coincides with the game  $\Gamma$ . A best response against a strategy combination  $s \in \Delta$  in the game  $\Gamma^t$  corresponds to a best response against the probability distribution  $t[s] + (1 - t)[p]$  in the game  $\Gamma$ . The latter probability distribution does not necessarily belong to  $\Delta$ , since it might not be an uncorrelated strategy combination.

The linear tracing procedure links a Nash equilibrium to the game  $\Gamma^0$ , i.e. a mixed strategy combination consisting of best responses against the prior beliefs, to a Nash equilibrium of  $\Gamma^1$ . Let  $\mathcal{L}$  denote the set of all Nash equilibria related to the games  $\Gamma^t$ ,  $t \in [0, 1]$ , so

$$\mathcal{L} = \{(t, s) \in [0, 1] \times \Delta \mid s \in NE(\Gamma^t)\}.$$

The linear tracing procedure is said to be feasible if there exists a continuous function  $\gamma : [0, 1] \rightarrow \mathcal{L}$ , i.e. a path, such that  $\gamma(0) \in \mathcal{L} \cap (\{0\} \times \Delta)$  and  $\gamma(1) \in \mathcal{L} \cap (\{1\} \times \Delta)$ .

Since  $\mathcal{L}$  is a set that can be described by a finite number of polynomial inequalities, it is a semi-algebraic set and all its components, i.e. maximally connected subsets, are also path-connected, so any two points in a component can be joined by a path (see Schanuel, Simon, and Zame (1991)). Therefore, to show that the linear tracing procedure is feasible, it is sufficient to show that  $\mathcal{L}$  has a component that intersects both the sets  $\{0\} \times \Delta$  and  $\{1\} \times \Delta$ .

The set  $\mathcal{L}^0 = \mathcal{L} \cap (\{0\} \times \Delta)$  is given by the Cartesian product over all players of the convex hull of the player's pure best responses to the prior. As a convex set, it is clearly connected, so it belongs to any component of  $\mathcal{L}$  that intersects the set  $\{0\} \times \Delta$ . Therefore, there is exactly one component of  $\mathcal{L}$  that intersects  $\{0\} \times \Delta$ . This component is denoted by  $\mathcal{L}^c$ . It remains to be shown that  $\mathcal{L}^c$  has a non-empty intersection with the set  $\{1\} \times \Delta$ .

### 3 Two proofs of feasibility of the linear tracing procedure

The key observation to give a simple proof of the feasibility of the linear tracing procedure is that one can use Browder's fixed point theorem as formulated in Browder (1960).

**Theorem 3.1 (Browder's fixed point theorem)** *Let  $S$  be a non-empty, compact, convex subset of  $\mathbb{R}^m$  and let  $f : [0, 1] \times S \rightarrow S$  be a continuous function. Then the set  $F = \{(\lambda, x) \in [0, 1] \times S \mid x = f(\lambda, x)\}$  contains a connected set  $F^c$  such that  $(\{0\} \times S) \cap F^c \neq \emptyset$  and  $(\{1\} \times S) \cap F^c \neq \emptyset$ .*

Our second proof will avoid working with correspondences and uses only Theorem 3.1. For the first proof we need the following generalization of Theorem 3.1, which is a special case of a result proved in Mas-Colell (1974).

**Theorem 3.2 (Mas-Colell's fixed point theorem)** *Let  $S$  be a non-empty, compact, convex subset of  $\mathbb{R}^m$  and let  $\varphi : [0, 1] \times S \rightarrow S$  be an upper hemi-continuous correspondence. Then the set  $F = \{(\lambda, x) \in [0, 1] \times S \mid x \in \varphi(\lambda, x)\}$  contains a connected set  $F^c$  such that  $(\{0\} \times S) \cap F^c \neq \emptyset$  and  $(\{1\} \times S) \cap F^c \neq \emptyset$ .*

Both Theorem 3.1 and Theorem 3.2 have a straightforward geometric intuition. For each  $\lambda \in [0, 1]$ , the existence of  $x$  such that  $x = f(\lambda, x)$  ( $x \in \varphi(\lambda, x)$ ) follows from Brouwer's fixed point theorem (Kakutani's fixed point theorem), which gives a continuum of fixed points. Theorems 3.1 and 3.2 point out that these fixed points have some additional structure. The faces  $\{0\} \times S$  and  $\{1\} \times S$  are joint by a connected subset of the set of fixed points. This property can be exploited immediately to show feasibility of the linear tracing procedure.

**Theorem 3.3** *Let  $\Gamma = (\Phi_1, \dots, \Phi_N, R_1, \dots, R_N)$  be a non-cooperative  $N$ -person game and let  $p \in \Delta$  be the prior. Then the linear tracing procedure is feasible.*

*Proof I.* Let the correspondence  $\beta_i : [0, 1] \times \Delta \rightarrow \Delta^{M_i}$  be defined by

$$\beta_i(t, s) = \{s_i^* \in \Delta^{M_i} \mid tR_i(s \setminus s_i^*) + (1-t)R_i(p \setminus s_i^*) \geq tR_i(s \setminus \bar{s}_i) + (1-t)R_i(p \setminus \bar{s}_i), \forall \bar{s}_i \in \Delta^{M_i}\}.$$

Since  $\beta_i$  is obtained by maximizing a continuous function in  $t$  and  $s$  over the compact set  $\Delta^{M_i}$  it follows by the maximum theorem that  $\beta_i$  is upper hemi-continuous. The correspondence  $\varphi : [0, 1] \times \Delta \rightarrow \Delta$  defined by  $\varphi(t, s) = \prod_{i \in N} \beta_i(t, s)$ ,  $\forall (t, s) \in [0, 1] \times \Delta$ , satisfies the conditions of Mas-Colell's fixed point theorem and so there is a component  $F^c$  of  $F = \{(t, s) \in [0, 1] \times \Delta \mid s \in \varphi(t, s)\}$  such that  $(\{0\} \times \Delta) \cap F^c \neq \emptyset$  and  $(\{1\} \times \Delta) \cap F^c \neq \emptyset$ . Obviously,  $F = \mathcal{L}$  and  $F^c = \mathcal{L}^c$ . Therefore,  $(\{1\} \times \Delta) \cap \mathcal{L}^c \neq \emptyset$  and the result follows. Q.E.D.

Combining the idea of the first proof with the approach used in Geanakoplos (1996), it is even possible to use Browder's fixed point theorem only and to avoid the use of correspondences. Geanakoplos's idea is to replace the maximizing principle used in the definition of  $\beta_i$  by the so-called satisficing principle. The satisficing principle means that instead of using the best response correspondence  $\beta_i$  it is sufficient to use a function that assigns to  $t$  and a mixed strategy combination  $s$  a mixed strategy that gives a player if possible a higher payoff than  $s_i$ , but not necessarily a maximal payoff.

*Proof II.* Let the function  $\sigma_i : [0, 1] \times \Delta \rightarrow \Delta^{M_i}$  be defined by

$$\sigma_i(t, s) = \arg \max_{\bar{s}_i \in \Delta^{M_i}} tR_i(s \setminus \bar{s}_i) + (1 - t)R_i(p \setminus \bar{s}_i) - \|\bar{s}_i - s_i\|^2,$$

where  $\|\cdot\|$  denotes the Euclidean norm. Obviously,  $\sigma_i$  is a continuous function since the penalty  $-\|\bar{s}_i - s_i\|^2$  is strictly concave in  $\bar{s}_i$ . The function  $f : [0, 1] \times \Delta \rightarrow \Delta$  defined by  $f(t, s) = (\sigma_1(t, s), \dots, \sigma_N(t, s))$ ,  $\forall (t, s) \in [0, 1] \times \Delta$ , satisfies the conditions of Browder's fixed point theorem and so there is a component  $F^c$  of  $F = \{(t, s) \in [0, 1] \times \Delta \mid s = f(t, s)\}$  such that  $(\{0\} \times \Delta) \cap F^c \neq \emptyset$  and  $(\{1\} \times \Delta) \cap F^c \neq \emptyset$ . Using exactly the same arguments as Geanakoplos (1996) we now show that  $F = \mathcal{L}$ . Clearly,  $\mathcal{L} \subset F$ . Now, suppose  $(\bar{t}, \bar{s}) \in F \setminus \mathcal{L}$ . Then, for some  $s_i \in \Delta^{M_i}$ ,  $H_i'(\bar{s} \setminus s_i) - H_i'(\bar{s}) = h > 0$ . Since  $H_i'(\bar{s} \setminus s_i) = \sum_{(i,k) \in \Phi_i} s_{ik} H_{ik}'(\bar{s} \setminus (i, k))$ , it holds that, for  $0 < \varepsilon < 1$ ,  $H_i'(\bar{s} \setminus \varepsilon s_i + (1 - \varepsilon)\bar{s}_i) - H_i'(\bar{s}) = \varepsilon h > 0$ . Now,  $\|(\varepsilon s_i + (1 - \varepsilon)\bar{s}_i) - \bar{s}_i\|^2 = \varepsilon^2 \|s_i - \bar{s}_i\|^2 < \varepsilon h$ , for small enough  $\varepsilon$ , contradicting that  $\bar{s}_i$  is the argument maximizing the expression in the definition of  $\sigma_i(\bar{t}, \bar{s})$ . Consequently,  $F = \mathcal{L}$ , so  $F^c = \mathcal{L}^c$ , and the result follows. Q.E.D.

Proof II implies that all major existence theorems in cooperative game theory, non-cooperative game theory, and general equilibrium theory can be shown without using correspondences and by means of constructions whose fixed points are the points of interest. The latter property is useful for computational purposes. Besides Brouwer's fixed point theorem, or results related to it like Browder's fixed point theorem, no heavy mathematical tools are needed. For the existence of core elements in an  $N$ -person cooperative non-transferable utility game such a construction is given in Herings (1997). For the existence of pure strategy Nash equilibria in non-cooperative  $N$ -person games with compact convex strategy sets<sup>2</sup>

<sup>2</sup> This contains the case of mixed strategy Nash equilibria for a finite non-cooperative  $N$ -person game as a special case.

and the existence of Walrasian equilibria in an Arrow-Debreu economy under assumptions similar to those of Debreu (1959), see Geanakoplos (1996). Finally, for the feasibility of the linear tracing procedure in non-cooperative  $N$ -person games, one can use the construction of the second proof.

## References

- Browder, F.E.: On continuity of fixed points under deformations of continuous mappings. *Summa Brasiliensis Mathematicae* 4, 183–191 (1960)
- van Damme, E.: Equilibrium selection in team games. In: Albers, W., Güth, W., Hammerstein, P., Moldovanu, B., van Damme, E. (eds.) *Understanding strategic behavior: essays in honor of Reinhard Selten*, pp. 100–111. Berlin Heidelberg New York: Springer 1996
- van Damme, E., Hurkens, S.: Games with imperfectly observable commitment. *Games and Economic Behavior* 21, 282–308 (1997)
- Debreu, G.: *Theory of value*. New Haven: Yale University Press 1959
- Geanakoplos, J.D.: Nash and Walras equilibrium via Brouwer. Cowles Foundation Discussion Paper No. 1131, New Haven, 1–18 (1996)
- Harsanyi, J.C.: The tracing procedure: a Bayesian approach to defining a solution for  $n$ -person noncooperative games. *International Journal of Game Theory* 4, 61–94 (1975)
- Harsanyi, J.C., Selten, R.: *A general theory of equilibrium selection in games*. Cambridge: MIT Press 1988
- Herings, P.J.J.: An extremely simple proof of the K-K-M-S theorem. *Economic Theory* 10, 361–367 (1997)
- Mas-Colell, A.: A note on a theorem of F. Browder. *Mathematical Programming* 6, 229–233 (1974)
- Schanuel, S.H., Simon, L.K., Zame, W.R.: The algebraic geometry of games and the tracing procedure. In: Selten, R. (ed.) *Game equilibrium models II: methods, morals and markets*, pp. 9–43. Berlin Heidelberg New York: Springer 1991

Copyright of Economic Theory is the property of Springer Science & Business Media B.V.. The copyright in an individual article may be maintained by the author in certain cases. Content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.