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The transition from a Drèze equilibrium to a Walrasian equilibrium [1]

Jean-Jacques Herings [a], Gerard van der Laan [b,*], Richard Venniker [b,2]

[a] *Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, Netherlands*

[b] *Department of Econometrics and Tinbergen Institute, Free University, De Boelelaan 1105, 1081 HV Amsterdam, Netherlands*

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Abstract

In this paper a price and quantity adjustment process in continuous time is considered for an economy facing price rigidities. In the short run prices are assumed to be completely fixed and the markets are cleared by quantity adjustments until a fixed price equilibrium is reached where every market is typically characterized by either supply rationing or demand rationing. Using only standard assumptions on the primitive concepts of the economy and a non-degeneracy condition, it is shown that the process indeed converges to a fixed price equilibrium for the initially given prices in the short run. In the long run prices are assumed to move upwards in the case of demand rationing on a market and downwards when supply rationing occurs, while markets are kept in equilibrium by infinitesimal quantity adjustments. Again, under standard assumptions on the primitive concepts of the economy and a non-degeneracy condition, the process is shown to reach a Walrasian equilibrium in the long run. A simplicial algorithm has been developed to make the study of the price and

* Corresponding author.

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quantity adjustment process possible and the accuracy of this algorithm is discussed.

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1. Introduction

Probably the best-known process of price adjustment in an exchange economy is the Walrasian tatonnement process. In a situation where demand does not equal supply on the markets of some commodities, the price on any market is adjusted according to the sign of the excess demand. Furthermore, as long as a non-zero excess demand prevails on some market, no exchange of commodities takes place. Eventually, it is hoped, a situation of zero excess demand will be obtained such that trade can take place.

The Walrasian tatonnement process has several shortcomings. One of the drawbacks of the Walrasian tatonnement process is that convergence to a Walrasian equilibrium price system cannot be guaranteed in general. Examples of economies for which the Walrasian tatonnement process does not converge have been given by Scarf (1960). It has been shown by Saari and Simon (1978) that convergence of a process based on the excess demand function can only be guaranteed when it uses almost all of the information provided by the excess demand function and its first derivatives. A well-known process that satisfies this condition is the global Newton method by Smale (1976), which converges to an equilibrium price system when started at the boundary of the price space. The processes by van der Laan and Talman (1987a,b) and Kamiya (1990) not only use local information about the excess demand, but are also based on information of the starting price system. As a consequence, it can be shown that the process of Kamiya converges under standard assumptions plus a reasonably mild boundary condition on the total excess demand function for a generic starting price system (see Kamiya, 1990), while for every starting price system the process of van der Laan and Talman converges for a generic exchange economy satisfying the standard conditions (see Herings, 1997).

Even when convergence takes place, it will take some time before the equilibrium price system has been reached. Until then, demand is not equal to supply on every market and trade is excluded by the Walrasian tatonnement process, as well as the other processes described above. A related drawback is that the relevant market signals for an adjustment process in an economy are based on the effective demand associated with a Drèze equilibrium (introduced by Drèze, 1975), as has been noticed in Veendorp (1975), and not on the notional demand used in the processes above. In a model with three commodities and two consumers and with the total excess demand function satisfying a gross substitutability condition, Veendorp has shown the convergence of an adjustment process that is based on effective excess demands and which follows the path of (unique) Drèze equilibria (see also Laroque, 1981, for a correction of the convergence proof). In general, however, such a process does not necessarily converge to a Walrasian equilibrium price system and even chaotic behaviour may be expected as argued in Day and Pianigiani (1991) and confirmed in a more complicated model in Böhm (1993). A model closely related to the one of Veendorp is considered in Movshovich (1994). Movshovich considers a process in discrete time, where at each point in time the price of a randomly chosen commodity is adjusted on the basis of the effective excess demand in the previous period at the (unique) Drèze equilibrium at the current price system. Also in the model of Movshovich gross substitutability like conditions are needed in order to guarantee convergence.

In Herings et al. (1997) an adjustment process is proposed such that a Drèze equilibrium prevails in the economy at any point generated by the process, and in which prices are adjusted according to the market situation at the Drèze equilibrium. That process is inspired by recent experiences in Eastern European countries and starts from a situation where the price level of the non-numeraire commodities is so low that complete demand rationing is necessary to equilibrate the markets. Next the process generates a path of Drèze equilibria in which only demand rationing prevails, until a Walrasian equilibrium is reached. Contrary to other price adjustment processes based on effective excess demands, this process converges to a Walrasian equilibrium price system under standard assumptions with respect to the economy.

The above described process only allows for initial situations in which there is demand rationing on all markets of the non-numeraire commodities. This implies that the process has two disadvantages. First, the process has to start from an initial situation at which the prices of the non-numeraire commodities are sufficiently low. Second, the process fails in explaining adjustments along a path of equilibria with both demand and supply rationing. As argued already in van der Laan (1980), in Western economies supply rationing occurs much more frequently than demand rationing. Therefore in this paper a price and quantity adjustment process is presented that follows a path of Drèze equilibria along which both demand and supply rationing may occur.

The starting point of the process is a historically given price system and quantity constraints which may not constitute a Drèze equilibrium. Initially, only the quantity constraints are adjusted until a Drèze equilibrium with respect to the initial price system has been reached. It is referred to as the short-term quantity adjustment process. Next, the process proceeds by adjusting prices and rationing schemes simultaneously in such a way that any point on the path reached by the process is a Drèze equilibrium. This path of Drèze equilibria either ends with a second Drèze equilibrium with respect to the initial price system or with a Walrasian equilibrium. In the first case, the process proceeds with a short-term quantity adjustment process, until a third Drèze equilibrium with respect to the initial fixed price system has been reached. It will be shown that eventually a Drèze equilibrium at the initial price system will be reached from where the path of Drèze equilibria leads to a Walrasian equilibrium. This part of the path is referred to as the long-term adjustment process. It should be noticed that the long-term process generates a path of Drèze equilibria and therefore at any point on this path trades can be carried out. Since the economy deviates from the Walrasian equilibrium, trade takes place under rationing and therefore may differ from the notional demands and supplies, as is generally accepted by 'Keynesian' economists; see e.g. Gordon (1990), who considers it as the essential feature of Keynesian macroeconomics. Since our adjustment process does converge to a Walrasian equilibrium, it gives some underpinning for the view, subscribed to by e.g. Mankiw (1994), that the economy does not stay away from the Walrasian equilibrium in the long run.

The adjustments made, in economic terms, are as follows. In the short-term process, prices do not adjust (this is based on the paradigm that quantity adjustments take place much faster than price adjustments) and, compared with the initial situation, supply rationing (demand rationing) adjusts such that it is maximal (minimal) in case of excess supply in a market, and minimal (maximal) in case of excess demand. Moreover, the rationing schemes adjust in such a way that markets in equilibrium stay in equilibrium, unless this would violate the maximality and minimality properties of rationing schemes mentioned before. In that case markets may get out of equilibrium. In the long-term process, the quantities adjust such that equality of constrained supply and demand always holds. Now prices adjust in such a way that, compared with the initial situation, they are minimal in case of supply rationing in a market and maximal in case of demand rationing in a market. Moreover, prices adjust in such a way that markets without rationing stay in equilibrium without rationing, again unless this would violate the maximality and minimality properties just mentioned. In that case, supply rationing or demand rationing may emerge again.

The convergence proof of the adjustment process described in this paper is based on some techniques of simplicial approximation, as initiated by Scarf (1973). First, an artificial function of excess demands, called the reduced total excess demand function, is constructed. The points reached by the adjustment process are zero points of this function. Subsequently, a simplicial algorithm is described and is shown to generate a path of zero points for a piecewise linear approximation of the reduced total excess demand function. This path is then shown to yield an approximation of the desired adjustment process. Similar to the processes of van der Laan and Talman (1987a,b) and Kamiya (1990), the adjustment process of this paper does not only use local information about the reduced total excess demand function, but is also based on information of the historically given price system and quantity constraints. Moreover, we derive a number of boundary properties of the reduced total excess demand function, that will guarantee that the adjustment process is bounded. These ingredients together with the specific definitions of the adjustments made, will be the main reasons for convergence to a Walrasian equilibrium.

2. The model

For ease of notation, in the sequel we denote the set of indices $\{1, \ldots, k\}$ by I_k, the set of indices $\{0, 1, \ldots, k\}$ by I_k^0, $\mathbb{R}_+^k = \{x \in \mathbb{R}^k | x_j \geq 0, \forall j \in I_k\}$, and $\mathbb{R}_{++}^k = \{x \in \mathbb{R}^k | x_j > 0, \forall j \in I_k\}$. We consider an exchange economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$. In this economy there are m consumers, indexed $i = 1, \ldots, m$, and $n+1$ commodities, indexed $j = 1, \ldots, n+1$. Each consumer $i \in I_m$ is characterized by a consumption set X^i, a preference preordering $\succeq^i$ on X^i, and a vector of initial endowments w^i. The vector w is defined by $w = \sum_{i \in I_m} w^i$. We assume that the economy $\mathscr{E}$ is initially faced with a completely fixed price system, determined by the vector $\tilde{p} \in \mathbb{R}_{++}^{n+1}$. We take one of the commodities, say commodity $n+1$, as the numeraire commodity having a price equal to 1, so $\tilde{p}_{n+1} = 1$. In the short run the prices are completely fixed and market equilibrium will be achieved by quantity constraints on the non-numeraire commodities. In the long run we allow for price flexibility by introducing a so-called 'flexibility' parameter $\alpha \in [0, 1)$, describing the maximal possible decrease or increase of a price relative to $\tilde{p}$. For a given vector $\tilde{p}$ of fixed prices and a flexibility parameter α, the set of admissible price systems is given by the set $P(\alpha)$ defined by

$$P(\alpha) = \left\{p \in \mathbb{R}_+^{n+1} | (1-\alpha)\tilde{p}_j \leq p_j \leq (1-\alpha)^{-1}\tilde{p}_j, \forall j \in I_n, p_{n+1} = 1\right\}$$

For ease of notation, in the following we define, for $\alpha \in [0, 1)$, $\underline{p}_j(\alpha) = (1-\alpha)\tilde{p}_j$, $\forall j \in I_n$, $\bar{p}_j(\alpha) = (1-\alpha)^{-1}\tilde{p}_j$, $\forall j \in I_n$, and $\underline{p}_{n+1}(\alpha) = \bar{p}_{n+1}(\alpha) = 1$. Clearly, the set of admissible price systems varies from the unique vector of fixed prices $\tilde{p}$ for $\alpha = 0$ to the set of all positive prices with the price of the numeraire normalized to one for $\alpha = 1$.

The following assumptions with respect to the economy $\mathscr{E}$ are made:

Assumption 1. For every consumer $i \in I_m$, the consumption set X^i belongs to $\mathbb{R}_+^{n+1}$, is closed and convex, and $X^i + \mathbb{R}_+^{n+1} \subset X^i$.

Assumption 2. For every consumer $i \in I_m$, the preference preordering $\succeq^i$ on X^i is complete, continuous, strongly monotonic, and strictly convex.

Assumption 3. For every consumer $i \in I_m$, the vector of initial endowments w^i belongs to the interior of X^i.

A preference preordering $\succeq^i$ is said to be strongly monotonic if $\bar{x}^i$, $\hat{x}^i \in X^i$, $\bar{x}^i \leq \hat{x}^i$, and $\bar{x}^i \neq \hat{x}^i$ implies $\hat{x}^i \succ \bar{x}^i$. A preference preordering $\succeq^i$ is said to be strictly convex if $\bar{x}^i$, $\hat{x}^i \in X^i$, $\bar{x}^i \neq \hat{x}^i$, $\bar{x}^i \sim^i \hat{x}^i$, and $\lambda \bar{x}^i + (1-\lambda)\hat{x}^i \in X^i$ for some $\lambda \in (0, 1)$ implies $\lambda \bar{x}^i + (1-\lambda)\hat{x}^i \succ^i \bar{x}^i$. Notice that the assumption of strict convexity in Assumption 2 allows us to work with demand functions instead of demand correspondences.

In general the short-term vector $\tilde{p}$ of fixed prices will not be equal to the price system in any Walrasian equilibrium of the economy $\mathscr{E}$, being a price system $p^* \in \mathbb{R}^{n+1}$ and consumption bundles $x^{*i} \in X^i$, $\forall i \in I_m$, such that both $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$ and x^{*i} is a best element for $\succeq^i$ in the budget set $\{x^i \in X^i | p^* \cdot x^i \leq p^* \cdot w^i\}$ for every $i \in I_m$. To equilibrate the demand and the supply under price rigidities one may introduce an equilibrium concept involving vectors of quantity constraints on the net demands and the net supplies of the non-numeraire commodities. Given a price system $p \in \mathbb{R}_+^{n+1}$, a rationing scheme on supply $l \in -\mathbb{R}_+^n$, and a rationing scheme on demand $L \in \mathbb{R}_+^n$, the constrained budget set of consumer $i \in I_m$ is given by

$$B^i(p,l,L) = \left\{x^i \in X^i | p \cdot x^i \leq p \cdot w^i \text{ and } l_j \leq x_j^i - w_j^i \leq L_j, \forall j \in I_n\right\}.$$

Following the fix-price literature (see for a recent survey Bénassy, 1993), it is assumed that there is no rationing on the market of the numeraire commodity. The corresponding constrained demand $d^i(p, l, L)$ of consumer i is defined as the best element for $\succeq^i$ in $B^i(p, l, L)$. Because of the strict convexity and strong monotonicity assumptions this element is unique and lies on the budget hyperplane, i.e. $p \cdot d^i(p, l, L) = p \cdot w^i$.

Following Drèze (1975), for given $\alpha \in [0, 1)$, a Drèze equilibrium with respect to the set $P(\alpha)$ of admissible price systems is defined as follows.

Definition 1 (*Drèze equilibrium*). For given $\alpha \in [0, 1)$, a Drèze equilibrium with respfect to the set $P(\alpha)$ (denoted DE_α) for the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ is a price system $p^* \in P(\alpha)$, a rationing scheme $(l^*, L^*) \in -\mathbb{R}_+^n \times \mathbb{R}_+^n$, and, for every consumer $i \in I_m$, a consumption bundle $x^{*i} \in X^i$ such that

1. for all $i \in I_m$, $x^{*i} = d^i(p^*, l^*, L^*)$;
2. $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$;
3. for all $j \in I_n$: $x_j^{*h} - w_j^h = l_j^*$ for some $h \in I_m$ implies $x_j^{*i} - w_j^i < L_j^*$, $\forall i \in I_m$, and, analogously, $x_j^{*h} - w_j^h = L_j^*$ for some $h \in I_m$ implies $x_j^{*i} - w_j^i > l_j^*$, $\forall i \in I_m$;
4. for all $j \in I_n$, $p_j^* > \underline{p}_j(\alpha)$ implies $l_j^* < x_j^{*i} - w_j^i$, $\forall i \in I_m$, and $p_j^* < \bar{p}_j(\alpha)$ implies $L_j^* > x_j^{*i} - w_j^i$, $\forall i \in I_m$.

The rationing schemes on supply and demand are assumed to be uniform, i.e. the same for each consumer. This assumption can be relaxed easily. Condition 1 requires that the consumption of each consumer equals his constrained demand, while Condition 2 is the market clearing condition. Condition 3 implies that there is not simultaneously rationing on both sides of a market. Condition 4 precludes supply rationing on the market of some commodity as long as its price is not minimal, whereas demand rationing on the market of a commodity does not take place if its price is not maximal. In case $\alpha = 0$, Condition 4 is redundant. For $\alpha > 0$, Condition 3 is implied by Condition 4.

A Drèze equilibrium without rationing yields a Walrasian equilibrium. It will be shown in Lemma 3 that for α large enough it holds that any DE_α is a Walrasian equilibrium.

3. The reduced total excess demand function

The price and quantity adjustment process will start with the vector of fixed prices $\tilde{p}$ and an arbitrary rationing scheme $(l, L) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$. In a first stage of quantity adjustments, the initial rationing scheme is adjusted until a short-term DE_0 for the vector of fixed prices $\tilde{p}$ is reached. In the second stage prices are adjusted while markets are kept in equilibrium by infinitesimally small changes in the rationing scheme. In this way the process will generate a path of DE_α for varying values of α and will finally converge to a Walrasian equilibrium.

Condition 4 of Definition 1 shows that for a Drèze equilibrium with respect to $P(\alpha)$ the price and the rationing scheme in a market have to satisfy the complementarity condition that rationing is not binding as long as the corresponding price is not on its upper or lower level. Therefore, to equilibrate markets, prices adjust as long as they are not on their upper or lower bound, while otherwise the rationing schemes are adjusted. Due to these complementarity conditions, it is possible to describe the price and rationing scheme for a commodity by one variable. The excess demands at the prices and rationing schemes defined by these variables are given by the so-called reduced total excess demand function. The set of variables defining the prices and rationing schemes is taken to be the $(n+1)$-dimensional set C^{n+1} given by

$$C^{n+1} = \left\{ q \in \mathbb{R}^{n+1} | 0 \le q_j \le 1, \forall j \in I_n, 0 \le q_{n+1} < 1 \right\}$$

To obtain the reduced total excess demand function, we relate to any element q of this set a flexibility parameter $\hat{\alpha}(q)$ and thereby a set $P(\hat{\alpha}(q))$ of admissible price systems, a price system $\hat{p}(q) \in P(\hat{\alpha}(q))$ and a rationing scheme $(\hat{l}(q), \hat{L}(q))$, defined by

$$\hat{\alpha}(q) = q_{n+1} \tag{1}$$

$$\hat{p}_{n+1}(q) = 1 \tag{2}$$

for all $j \in I_n$,

$$\hat{p}_j(q) = \max\{\underline{p}_j(\hat{\alpha}(q)), \min\{(2 - 3q_j)\underline{p}_j(\hat{\alpha}(q)) + (3q_j - 1)\bar{p}_j(\hat{\alpha}(q)), \bar{p}_j(\hat{\alpha}(q))\}\} \tag{3}$$

$$\hat{l}_j(q) = -\min\{1, 3q_j\}w_j \tag{4}$$

$$\hat{L}_j(q) = \min\{1, 3 - 3q_j\}\frac{\tilde{p} \cdot w}{\tilde{p}_j}. \tag{5}$$

Let some $q \in C^{n+1}$ be given. For every $j \in I_n$ we have that $q_j = 0$ implies $\hat{l}_j(q) = 0$ and $q_j = 1$ implies $\hat{L}_j(q) = 0$. Moreover, $0 \leq q_j < 1/3$ implies $\hat{p}_j(q) = \underline{p}_j(\hat{\alpha}(q))$, $\hat{l}_j(q) > -w_j$, and $\hat{L}_j(q) = \tilde{p} \cdot w/\tilde{p}_j$. Furthermore $1/3 \leq q_j \leq 2/3$ implies $\underline{p}_j(\hat{\alpha}(q)) \leq \hat{p}_j(q) \leq \bar{p}_j(\hat{\alpha}(q))$, $\hat{l}_j(q) = -w_j$, and $\hat{L}_j(q) = \tilde{p} \cdot w/\tilde{p}_j$, whereas $2/3 < q_j \leq 1$ implies $\hat{p}_j(q) = \bar{p}_j(\hat{\alpha}(q))$, $\hat{l}_j(q) = -w_j$, and $\hat{L}_j(q) < \tilde{p} \cdot w/\tilde{p}_j$. For any q it holds that $\hat{p}(q) \in P(\hat{\alpha}(q))$. So, supply rationing prevails at the minimum admissible price when $0 \leq q_j \leq 1/3$. When $1/3 \leq q_j \leq 2/3$, then neither supply rationing nor demand rationing exists. Finally, when $2/3 \leq q_j \leq 1$, there may exist demand rationing, there is no rationing on the supply side, and the price is on its upper bound. For $q \in C^{n+1}$ we call $\hat{B}^i(q) = B^i(\hat{p}(q), \hat{l}(q), \hat{L}(q))$ the reduced constrained budget set of consumer $i \in I_m$ at q, i.e.

$$\hat{B}^i(q) = \{x^i \in X^i | \hat{p}(q) \cdot x^i \leq \hat{p}(q) \cdot w^i.$$

and

$$\hat{l}_j(q) \leq x^i_j - w^i_j \leq \hat{L}_j(q), \forall j \in I_n\}$$

Let $\hat{d}^i(q)$ denote the best element for $\succeq^i$ in the reduced constrained budget set $\hat{B}^i(q)$ of consumer $i \in I_m$, so $\hat{d}^i(q) = d^i(\hat{p}(q), \hat{l}(q), \hat{L}(q))$, let $\hat{d}(q) = (\hat{d}^1(q), \ldots, \hat{d}^m(q))$ denote the collection of vectors of resulting demands, and define the total excess demand at q by

$$\hat{z}(q) = \sum_{i=1}^{m} \hat{d}^i(q) - \sum_{i=1}^{m} w^i.$$

The function $\hat{z}$: $C^{n+1} \to \mathbb{R}^{n+1}$ is called the reduced total excess demand function. For $q^* \in C^{n+1}$ it holds that $(\hat{p}(q^*), \hat{l}(q^*), \hat{L}(q^*), \hat{d}(q^*))$ is a $\text{DE}_{\hat{\alpha}(q^*)}$ if and only if $\hat{z}(q^*) = \underline{0}$. The price and quantity adjustment process can now be described in terms of q and the total excess demand induced by q, $\hat{z}(q)$, only.

Before we describe the process in detail, we describe some properties of the reduced total excess demand function $\hat{z}$ in the following lemmas.

Lemma 1. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ *satisfy Assumptions* 1–3. *Then the reduced total excess demand function* $\hat{z}$ *is continuous on* C^{n+1} *and* $\hat{p}(q) \cdot \hat{z}(q) = 0$, $\forall q \in C^{n+1}$.

Proof. By the lemma in Drèze (1975, p. 304) it follows that, for every $i \in I_m$, B^i is continuous on $\mathbb{R}^n_+ \times \{1\} \times -\mathbb{R}^n_+ \times \mathbb{R}^n_+$, using that $p_{n+1} = 1$ and the absence of rationing on the market of the numeraire commodity. Using the continuity and the strict convexity of the preferences and the maximum theorem it follows that, for every $i \in I_m$, d^i is continuous on $\mathbb{R}^n_+ \times \{1\} \times -\mathbb{R}^n_+ \times \mathbb{R}^n_+$. By the continuity of the functions $\hat{\alpha}$, $\hat{p}$, $\hat{l}$ and $\hat{L}$ in q it follows that $\hat{z}$ is continuous on C^{n+1}. The strong monotonicity of the preferences yields that

$$\hat{p}(q) \cdot \hat{z}(q) = 0, \forall q \in C^{n+1}. \qquad \text{QED}$$

The next lemma gives the behaviour of $\hat{z}_j(q)$ on C^{n+1} for values of q_j, $j \in I_n$, equal to zero or one.

Lemma 2. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ satisfy Assumptions 1–3. *Then, for every $q \in C^{n+1}$, for every $j \in I_n$, $q_j = 0$ implies $\hat{z}_j(q) \geq 0$ and $q_j = 1$ implies $\hat{z}_j(q) \leq 0$.*

Proof. If $q_j = 0$ for some $j \in I_n$, then $\hat{l}_j(q) = 0$ and hence $\hat{z}_j(q) = \sum_{i=1}^m (\hat{d}_j^i(q) - w_j^i) \geq m\hat{l}_j(q) = 0$. Analogously, if $q_j = 1$ for some $j \in I_n$, then $\hat{L}_j(q) = 0$, and hence $\hat{z}_j(q) = \sum_{i=1}^m (\hat{d}_j^i(q) - w_j^i) \leq m\hat{L}_j(q) = 0$. QED

Lemma 2 will be used to show that the adjustment process will not hit the boundary of C^{n+1} where $q_j = 0$ or $q_j = 1$ for some $j \in I_n$.

If $q^* \in C^{n+1}$ satisfies $q^*_{n+1} = 0$ and $\hat{z}(q^*) = \underline{0}$, then $\hat{p}(q^*) = \tilde{p}$ and $(\hat{p}(q^*), \hat{l}(q^*), \hat{L}(q^*), \hat{d}(q^*))$ is a DE_0. The next lemma states that if $q^* \in C^{n+1}$ is such that $\hat{\alpha}(q^*) = q^*_{n+1}$ is sufficiently close to one and $\hat{z}(q^*) = \underline{0}$ then $(\hat{p}(q^*), \hat{l}(q^*), \hat{L}(q^*), \hat{d}(q^*))$ is a Walrasian equilibrium, i.e. for every consumer $i \in I_m$ it holds that $\hat{l}_j(q^*) < \hat{d}_j^i(q^*) - w_j^i < \hat{L}_j(q^*)$.

Lemma 3. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ satisfy Assumptions 1–3. *Then there exists $\beta \in (0, 1)$ such that for every $q^* \in C^{n+1}$ satisfying $\hat{z}(q^*) = \underline{0}$ and $q^*_{n+1} \geq \beta$ it holds that $(\hat{p}(q^*), \hat{l}(q^*), \hat{L}(q^*), \hat{d}(q^*))$ is a* Walrasian equilibrium.

Proof. Suppose there exists $\alpha^* \in (0, 1)$ such that $q^*_{n+1} < \alpha^*$ for any zero point $q^* \in C^{n+1}$ of $\hat{z}$. Then the lemma is true for $\beta = \alpha^*$.

Now, suppose the lemma is not true. Then there is a sequence $(q^r)_{r \in \mathbb{N}}$ such that for every $r \in \mathbb{N}$, $q^r_{n+1} \leq r - 1/r$, $\hat{z}(q^r) = \underline{0}$ and $(\hat{p}(q^r), \hat{l}(q^r), \hat{L}(q^r), \hat{d}(q^r))$ is a $\text{DE}_{\hat{\alpha}(q^r)}$ but not a Walrasian equilibrium. Modify the consumption set X^i, $\forall i \in I_m$, by imposing the constraint $x^i_{n+1} \leq w^i_{n+1} + w_{n+1}$. Since q^r induces a $\text{DE}_{\hat{\alpha}(q^r)}$, $\hat{d}^i(q^r)$, $\forall i \in I_m$, satisfies this constraint with inequality. Without loss of generality, it can be assumed that (i) there is a commodity $j' \in I_n$ such that $\hat{p}_{j'}(q^r) = \bar{p}_{j'}(\hat{\alpha}(q^r))$, $\forall r \in N$, or (ii) there is a commodity $j' \in I_n$ such that $\hat{p}_{j'}(q^r) = \underline{p}_{j'}(\hat{\alpha}(q^r))$, $\forall r \in N$, since otherwise there could be no binding rationing in the $\mathrm{DE}_{\hat{\alpha}(q^r)}$ induced by q^r by Condition 4, implying that q^r yields a Walrasian equilibrium, contradicting our supposition. Consider the sequence

$$\left(\frac{\hat{p}(q^r)}{\|\hat{p}(q^r)\|_\infty}, \hat{l}(q^r), \hat{L}(q^r), \hat{d}(q^r) \right)_{r \in \mathbb{N}},$$

which is bounded and therefore, without loss of generality, can be assumed to converge to some $(p', l', L', d') \in \mathbb{R}^{n+1}_+ \times -\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \prod_{i=1}^m \mathbb{R}^{n+1}_+$. Notice that Condition 2 of Definition 1 implies that $\sum_{i \in I_m} d'^i = \sum_{i \in I_m} w^i$. Clearly, there is $k \in I_{n+1}$ such that $p'_k = 1$. It follows easily that $l'_k = -w_k$ if $k \in I_n$. Therefore, even in case $k = n+1$, $p' \cdot (l'^{\mathrm{T}}, -w_{n+1})^{\mathrm{T}} < 0$, and so the function d^i, $\forall i \in I_m$, is continuous at (p', l', L') by the lemma in Drèze (1975, p. 304). In case (i) it holds that $p'_{n+1} = \lim_{r \to \infty} \hat{p}_{n+1}(q^r)/\|\hat{p}(q^r)\|_\infty = 0$, so by the strict monotonicity of preferences, for every $i \in I_m$,

$$\begin{aligned} d'^i_{n+1} &= \lim_{r \to \infty} \hat{d}^i_{n+1}(q^r) = \lim_{r \to \infty} d^i_{n+1}\big(\hat{p}(q^r), \hat{l}(q^r), \hat{L}(q^r)\big) \\ &= d^i_{n+1}(p', l', L') = w^i_{n+1} + w_{n+1}, \end{aligned}$$

so $w_{n+1} = \sum_{i \in I_m} d'^i_{n+1} = (m+1) w_{n+1}$, a contradiction. In case (ii) it holds that $p'_{j'} = 0$ and $L'_{j'} = \tilde{p} \cdot w / \tilde{p}_{j'}$, so by the strict monotonicity of preferences, for every $i \in I_m$,

$$\begin{aligned} d'^i_{j'} &= \lim_{r \to \infty} \hat{d}^i_{j'}(q^r) = \lim_{r \to \infty} d^i_{j'}\big(\hat{p}(q^r), \hat{l}(q^r), \hat{L}(q^r)\big) \\ &= d^i_{j'}(p', l', L') = w^i_{j'} + \frac{\tilde{p} \cdot w}{\tilde{p}_{j'}} > w^i_{j'} + w_{j'}, \end{aligned}$$

so $w_{j'} = \sum_{i \in I_m} d'^i_{j'} > (m+1) w_{j'}$, a contradiction. Consequently, the lemma is true. QED

For the remainder of the paper we fix a real number $\bar{\beta} \in (0, 1)$, satisfying the requirement of Lemma 3, and define the sets $\hat{C}^{n+1}$, $\underline{C}^{n+1}$, and $\bar{C}^{n+1}$ by $\hat{C}^{n+1} = \{q \in C^{n+1} | q_{n+1} \le \bar{\beta}\}$, $\underline{C}_{n+1} = \{q \in C^{n+1} | q_{n+1} = 0\}$ and $\bar{C}^{n+1} = \{q \in C^{n+1} | q_{n+1} = \bar{\beta}\}$. In Section 5 we give a constructive proof of the existence of a path of points in $\hat{C}^{n+1}$ leading from an arbitrarily chosen initial state $v \in \underline{C}_{n+1}$ to an approximate zero point $q^* \in \bar{C}^{n+1}$ of $\hat{z}$ if some non-degeneracy condition is satisfied. Since q^* is an element of $\bar{C}^{n+1}$, it induces an approximate Walrasian equilibrium. The path itself corresponds to the desired adjustment process for a piecewise linear approximation of the reduced total excess demand function, where the inaccuracy of the approximation can be made arbitrarily small.

4. Definition of the price and quantity adjustment process

In this section an initial state v in the relative interior of $\underline{C}^{n+1}$ is assumed to be given. Notice that the initial state v induces a flexibility parameter $\hat{\alpha}(v)=0$, a price system $\hat{p}(v)=\tilde{p}$, and a rationing scheme $(\hat{l}(v),\ \hat{L}(v))$. The price and quantity adjustment process consists of two parts. First, only quantities, i.e. the rationing schemes, are adjusted and the price system is assumed to be fixed and equal to $\tilde{p}$. This part of the adjustment process is referred to as the short-term adjustment process. The short-term adjustment process results in a DE_0 and coincides with the quantity adjustment process given in Herings (1994) to find a Drèze equilibrium given a fixed price system. Secondly, after a DE_0 has been found, the prices are adjusted and the quantities, i.e. the rationing schemes, are assumed to change in such a way that the markets stay in equilibrium. This part of the adjustment process is referred to as the long-term adjustment process and results in a Walrasian equilibrium of the economy. The long-term path can be seen as an adjustment process in continuous time leading to a Walrasian equilibrium and having an attractive economic interpretation, namely that at any point in time we have a Drèze equilibrium with respect to the current set of admissible prices. So, keeping all markets in equilibrium by a rationing scheme satisfying the Drèze complementarity conditions while expanding the set of admissible prices from the single initially given price vector $\tilde{p}$ to the set of all non-negative prices provides us with a convergent process leading to a Walrasian equilibrium and allowing trade at any point in time.

More in detail, the behaviour of the price and quantity adjustment process can be described as follows. Initially, from v the path proceeds in the subset $\underline{C}^{n+1}$ of $\hat{C}^{n+1}$ until a zero point $q^* \in \underline{C}^{n+1}$ of $\hat{z}$ is reached. Clearly, q^* induces a DE_0 since at q^* it holds that $\hat{\alpha}(q^*)=q^*_{n+1}=0$. In general, the initial state v is incompatible with a DE_0, i.e. $\hat{z}(v)\neq \underline{0}$. First only adjustments of the rationing scheme take place, being based on the excess demands on the markets of the non-numeraire commodities and on the change of the rationing scheme compared with the initial state, whereas the price system remains initially unchanged. So initially the process generates a path in $\underline{C}^{n+1}$. If there is a negative excess demand on a market, the rationing scheme is adjusted in such a way that, compared with the initial state induced by point v, the rationing is tightened maximally in case of supply rationing and is weakened maximally in case of demand rationing. So, compared with the initial state, consumers are enforced to supply less or are allowed to demand more of this commodity. If there is a positive excess demand on a market, the rationing scheme is adjusted in such a way that, compared with the initial state induced by point v, rationing is tightened maximally in case of demand rationing and weakened maximally in case of supply rationing. If a market is in equilibrium, then the rationing scheme is adjusted in such a way that the market is kept in equilibrium. These adjustments seem natural in economic terms.

The properties of the states of the economy reached by this quantity adjustment process can be formulated mathematically as follows. If a point $q \in \overline{C}^{n+1}$ is reached by the process, then there exists a real number $\gamma \in [0, 1]$ such that for every $j \in I_n$,

$$\begin{aligned} q_j &= \gamma v_j && \text{if } \hat{z}_j(q) < 0 \\ \gamma v_j \leq q_j &\leq 1 - \gamma(1 - v_j) && \text{if } \hat{z}_j(q) = 0 \\ q_j &= 1 - \gamma(1 - v_j) && \text{if } \hat{z}_j(q) > 0. \end{aligned} \tag{6}$$

Notice that the initial state v satisfies the properties given above for $\gamma = 1$. The adjustment process then proceeds by decreasing γ, i.e. decreasing q_j if there is excess supply on the market of commodity j, and increasing q_j if there is excess demand on the market of commodity j. The process continues like this, until one of the markets of the non-numeraire commodities becomes in equilibrium, say the market of commodity j^1. Then q_{j^1} adjusts such that the market of commodity j^1 is kept in equilibrium, whereas q_js of markets in excess supply are kept minimal and q_js of markets in excess demand maximal. Furthermore, if there was excess supply in the market of commodity j^1 before it was equilibrated, then q_{j^1} is increased away from the minimum value, and if there was excess demand in the market of commodity j^1 before equilibration, then q_{j^1} is decreased away from the maximum value. Again, the process continues until one of the markets of commodities $j^2 \in I_n \setminus \{j^1\}$ is equilibrated, after which q_{j^2} adjusts such that it stays in equilibrium, or it holds that in order to keep the market of commodity j^1 in equilibrium, either q_{j^1} would fall below the minimum value γv_{j^1} or would rise above the maximum value $1 - \gamma(1 - v_{j^1})$. In the first case, q_{j^1} is kept equal to γv_{j^1} and excess supply results on the market of commodity j^1, and in the latter case q_{j^1} is kept equal to $1 - \gamma(1 - v_{j^1})$ and excess demand results on the market of commodity j^1. The process continues in this way until a Drèze equilibrium is reached.

Observe that the properties of the short-term adjustment process are closely related to the ideas behind the Walrasian tatonnement process, but now with adjustments of quantities instead of prices. In fact, at a point q reached by the process we have that $q_j / v_j = \min_{h \in I_n} \{q_h / v_h\}$ if $\hat{z}_j(q) < 0$, so that q_j is minimal relative to v, if there is excess supply of commodity j. On the other hand we have that $(1 - q_j)/(1 - v_j) = \min_{h \in I_n} \{(1 - q_h)/(1 - v_h)\}$ if $\hat{z}_j(q) > 0$, so that relatively $1 - q_j$ is minimal if there is excess demand of commodity j. Finally, when $\hat{z}_j(q) = 0$, then q_j is allowed to vary between these relative lower and upper bounds in order to keep market j in equilibrium.

In Herings (1994) it is shown that for a generic economy (parametrized by the initial endowments) the properties in Eq. (6) determine the short-term adjustment process, up to the speed of adjustment, uniquely, and that it converges to a uniquely determined DE_0. The proofs given there are for the function $\hat{z}$ itself, but are extremely tedious. Therefore, the proofs given in the next section are related to a piecewise linear approximation of $\hat{z}$, allowing us to give considerably easier demonstrations. As a heuristic for the former statement, notice that the system of equations in Eq. (6) has $n+1$ independent free variables, $q_1, \ldots, q_n$ and γ, and n equations, leaving 1 degree of freedom being the dimension of the adjustment path. The process leads to a point $q^* \in \underline{C}^{n+1}$, yielding a DE_0. Observe that at this point the properties in Eq. (6) are satisfied for $\gamma = \min_{j \in I_n}\{q_j^*/v_j, (1-q_j^*)/(1-v_j)\}$.

Continuing from this point q^*, the process generates a path of zero points of $\hat{z}$ in $\hat{C}^{n+1}$. Along the path the flexibility parameter α is adjusted according to the adjustment of the value of q_{n+1}. Moreover, if there is demand (supply) rationing on the market of a commodity, then its price is increased (decreased) relatively maximal, whereas the rationing scheme is adjusted such that any point on the path induces a $\mathrm{DE}_{\hat{\alpha}(q)}$. It will be shown that this long-term process of following a path of $\mathrm{DE}_{\hat{\alpha}(q)}$s by price and quantity adjustments either comes back to a point in $\underline{C}^{n+1}$ at which $q_{n+1} = 0$, or ends at a point in $\overline{C}^{n+1}$ at which $q_{n+1} = \overline{\beta}$. In the latter case a Walrasian equilibrium has been found according to Lemma 3. In the former case the process has found a second DE_0. Then the process continues in $\underline{C}^{n+1}$ by short-term adjustments of the rationing scheme determined by Eq. (6) until a new (third) DE_0 is found. From this point on the process continues again in $\hat{C}^{n+1}$ by following a path of $\mathrm{DE}_{\hat{\alpha}(q)}$s. It will be shown that eventually a DE_0 will be reached from where the long-term path of $\mathrm{DE}_{\hat{\alpha}(q)}$s leads to a point q in $\overline{C}^{n+1}$, i.e. to a Walrasian equilibrium.

Mathematically, the properties of the states of the economy reached by the long-term adjustment process can be formulated as follows. If $q^* \in \hat{C}^{n+1}$ is reached by the long-term adjustment process, then

$$\hat{z}(q^*) = \underline{0}, \tag{7}$$

so q^* induces a $\mathrm{DE}_{\hat{\alpha}(q^*)}$. Again, it should be expected that for a generic economy property Eq. (7) determines the long-term adjustment process, up to the speed of adjustment, uniquely. As a heuristic, notice that the system of equations in Eq. (7) has $n+1$ independent free variables, $q_1, \ldots, q_{n+1}$, and, by Walras law, n independent equations, leaving 1 degree of freedom being the dimension of the adjustment path. It can now also be made intuitively clear that finally a Walrasian equilibrium is obtained. All Drèze equilibria with respect to $\tilde{p}$ $(q_{n+1}^* = 0)$ satisfy Eq. (7) as well as all Walrasian equilibria $(q_{n+1}^* = \overline{\beta})$. It should be expected that, generically, Eq. (7) determines a one-dimensional manifold with boundary, with the Drèze and the Walrasian equilibria as boundary points. Moreover, it is known that generically there is both an odd number of Drèze equilibria and an odd number of Walrasian equilibria. Consider the first DE_0 found by the short-term adjustment process. It is the end-point of a path described by Eq. (7) having either another DE_0 as end point or a Walrasian equilibrium, after which the long-term adjustment process terminates. In the former case, since the number of DE_0 is odd, there exists another DE_0 not yet reached by the adjustment process. Such a DE_0 will be found by the short-term adjustment process. Again, this is the end point of a path described by Eq. (7) having yet another DE_0 or a Walrasian equilibrium as end point. Using the oddness of the Drèze equilibria with respect to $\tilde{p}$ it follows that finally a DE_0 is reached that leads to a Walrasian equilibrium.

By definition of the reduced total excess demand function $\hat{z}$, if holds along the path determined by property Eq. (7) that, for $j \in I_n$, the prices satisfy the conditions

$$\begin{aligned} &\hat{p}_j(q^*) = (1-\hat{\alpha}(q^*))\tilde{p}_j && \text{if } q_j^* \le \tfrac{1}{3} \\ &(1-\hat{\alpha}(q^*))\tilde{p}_j \le \hat{p}_j(q^*) \le (1-\hat{\alpha}(q^*))^{-1}\tilde{p}_j && \text{if } \tfrac{1}{3} \le q_j^* \le \tfrac{2}{3} \\ &\hat{p}_j(q^*) = (1-\hat{\alpha}(q^*))^{-1}\tilde{p}_j && \text{if } q_j^* \ge \tfrac{2}{3}. \end{aligned} \tag{8}$$

Therefore, along the path of long-term price and quantity adjustments, the price of a commodity is kept on its lower bound with respect to $\hat{\alpha}(q^*)$ as long as there is supply rationing on the market of this commodity and the price of a commodity is kept on its upper bound with respect to $\hat{\alpha}(q^*)$ as long as there is demand rationing on the market of this commodity. If there is no rationing on the market of a commodity, then its price may vary between these lower and upper bounds, to keep that market in equilibrium without rationing.

Notice that the properties mentioned above are again closely related to the ideas behind the Walrasian tatonnement process. Starting from q^* inducing a DE_0, the flexibility parameter α is increased, implying that prices of commodities in the market on which there is supply rationing are decreased maximally, while prices of commodities in the market on which there is demand rationing are increased maximally. In general, none of the markets is in equilibrium without rationing yet. The long-term adjustment process continues, until one of the markets, say market j^3, is cleared without rationing, or another DE_0 is found. In the latter case, the short-term adjustment process will yield a third DE_0, from which the long-term adjustment process continues. In the former case the price of the corresponding commodity adjusts such that the market is kept in equilibrium, implying that it is increased away from the minimum value $(1-\hat{\alpha}(q))\tilde{p}_{j^3}$ in the case there was supply rationing in this market before the equilibration, and the price is decreased away from the maximum value $(1-\hat{\alpha}(q))^{-1}\tilde{p}_{j^3}$ in the case there was demand rationing in this market. The prices of the other commodities are kept minimal in the case of supply rationing and maximal in the case of demand rationing. The long-term adjustment process continues like this until one of the markets in $I_n \setminus \{j^3\}$ is equilibrated without rationing, say the market of commodity j^4, or in order to keep market j^3 in equilibrium its price would fall below the minimum value $(1-\hat{\alpha}(q))\tilde{p}_{j^3}$ or would rise above $(1-\hat{\alpha}(q))^{-1}\tilde{p}_{j^3}$. In the latter case, the price of commodity j^3 is kept equal to this minimum or maximum value, and supply rationing is used in the first case and demand rationing in the second case, to clear the markets. In the former case, the price of commodity j^4 adjusts to keep this market in equilibrium without rationing, etc. All these properties are a simple consequence of Eq. (7) and the definition of the reduced total excess demand function $\hat{z}$.

Let Q denote the subset of $\hat{C}^{n+1}$ whose points satisfy Eq. (6) or Eq. (7), i.e. all points q satisfying the properties of points generated by the price and quantity adjustment process. As has been argued above the set Q can be expected in general to be a compact one-dimensional manifold with boundary, so a finite collection of paths and loops. Not all the points in the set Q will actually be reached by the adjustment process. This will only hold for the points in Q which are connected to the starting point v, i.e. which are on the path starting in v. Denote the component of v in Q by Q_v. This leads us to the following definition.

Definition 2 (price and quantity adjustment process). The *price and quantity adjustment process* for the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ with starting point $v \in \underline{C}^{n+1}$ is given by the set Q_v, the component of v in Q.

Notice that the price and quantity adjustment process is now defined by considering explicitly the points q generated by it. Since we have given a topological definition of the process, we will also give a topological definition of the convergence of it.

Definition 3 (convergence). The price and quantity adjustment process for the economy $\mathscr{E} = (\{X^i, \succeq^i, w_i\}_{i=1}^m, \tilde{p})$ with starting point $v \in \underline{C}^{n+1}$ is convergent if Q_v is an arc, i.e. a set homeomorphic to the unit interval [0, 1], having v and a Walrasian equilibrium price system as its boundary points.

Notice that any arc described as the zero points of a system of continuously differentiable functions satisfying certain regularity properties can be described by a system of differential equations, see for instance Garcia and Zangwill (1981). Hence, by making suitable differentiability and regularity conditions, it is possible to express the adjustment process in a more standard way by means of a system of differential equations. This is also the approach as adopted by Smale (1976) and Kamiya (1990). Such a system of differential equations will depend on the function $\hat{z}$ and its first derivatives (where these are well-defined), exactly the amount of information indicated by Saari and Simon (1978). In this paper we will not make any differentiability assumptions. However, we will provide a method to follow the adjustment process arbitrarily close, even in the case where the function $\hat{z}$ is only continuous.

5. The approximate price and quantity adjustment process

In this section attention is focused on the price and quantity adjustment process for a piecewise linear approximation of the reduced total excess demand function, where the inaccuracy of the approximation can be taken arbitrarily small. We show that, under a standard non-degeneracy condition, for any arbitrarily chosen starting point v in the relative interior of C^{n+1} there exists a unique path of points connecting v with a point in $\overline{C}^{n+1}$ yielding an approximate Walrasian equilibrium. So we show that the price and quantity adjustment process related to any piecewise linear approximation of the reduced total excess demand function satisfying a non-degeneracy condition, is convergent. Applying the technique of simplicial approximation provides us both with a constructive proof of the existence of such a path of points and with the possibility to follow this path. By taking the mesh size of the underlying triangulation small enough the inaccuracy of the approximation can be made arbitrarily small. In the following definition an approximate DE_α for any parameter value of $\alpha \in [0, 1)$ is introduced.

Definition 4 ($\varepsilon - \mathrm{DE}_\alpha$). For given $\alpha \in [0, 1)$ and given real number $\varepsilon \geq 0$, an $\varepsilon - \mathrm{DE}_\alpha$ for the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ is a price system $p \in P(\alpha)$, a rationing scheme $(l, L) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and for every consumer $i \in I_m$ a consumption bundle $x^i \in X^i$ such that all conditions of a DE_α are satisfied with the condition of equality of demand and supply replaced by $\|\sum_{i=1}^m x^i - \sum_{i=1}^m w^i\|_\infty \leq \varepsilon$.

Clearly, a $0 - \mathrm{DE}_\alpha$ is a DE_α. In order to show the existence of the described path we will use some techniques of simplicial approximation of functions. We first recall the concept of a simplicial subdivision.

For given $t \in \mathbb{N}$, $0 \leq t \leq k$, a t-dimensional simplex or t-simplex is defined as the convex hull of $t+1$ affinely independent vectors in $\mathbb{R}^k$, $q^1, \ldots, q^{t+1}$, and is denoted by $\sigma(q^1, \ldots, q^{t+1})$ or shortly by σ. The vectors $q^1, \ldots, q^{t+1}$ are called the vertices of σ. A $(t-1)$-simplex τ being the convex hull of t vertices of $\sigma(q^1, \ldots, q^{t+1})$ is called a facet of σ. For $h \in I_{t+1}$ the facet $\tau(q^1, \ldots, q^{h-1}, q^{h+1}, \ldots, q^{t+1})$ is called the facet of σ opposite the vertex q^h. For $0 \leq j \leq t$, a j-simplex being the convex hull of $j+1$ vertices of a t-simplex σ is called a face of σ. A finite collection $\mathscr{T}$ of k-simplices is a triangulation of a compact, convex k-dimensional subset S of some Euclidean space if:

1. S is the union of all simplices in $\mathscr{T}$;
2. the intersection of two simplices in $\mathscr{T}$ is either empty or a common face of both.

It can be shown that each facet τ of a k-simplex $\sigma \in \mathscr{T}$ either lies in the relative boundary of S and is only a facet of σ or it is a facet of exactly one other k-simplex in $\mathscr{T}$. The mesh size of a triangulation $\mathscr{T}$ is defined by $\mathrm{mesh}(\mathscr{T}) = \max_{\sigma \in \mathscr{T}} \max\{\|\tilde{q} - \hat{q}\|_\infty | \tilde{q}, \hat{q} \in \sigma\}$.

Let a compact, convex t-dimensional subset S of some Euclidean space, a triangulation $\mathscr{T}$ of S and a function f: $S \to \mathbb{R}^k$ be given. A function F: $S \to \mathbb{R}^k$ is called the piecewise linear approximation of f with respect to $\mathscr{T}$ if for every vertex q of any $\sigma \in \mathscr{T}$ it holds that $F(q) = f(q)$ and for every element q of S it holds that $F(q) = \sum_{h \in I_{t+1}} \lambda_h F(q^h)$, when $q \in \sigma(q^1, \ldots, q^{t+1})$ for some t-simplex $\sigma \in \mathscr{T}$ and $q = \sum_{h \in I_{t+1}} \lambda_h q^h$ for some $\lambda \in \mathbb{R}_+^{t+1}$ with $\sum_{h \in I_{t+1}} \lambda_h = 1$.

The price and quantity adjustment process will be considered for a piecewise linear approximation $\hat{Z}$ of the reduced total excess demand function $\hat{z}$ with respect to a given triangulation $\mathscr{T}$ of $\hat{C}^{n+1}$ with an arbitrarily small mesh size. As is intuitively clear, and as will be shown formally later on, a point $q \in \hat{C}^{n+1}$ satisfying $\hat{Z}(q) = \underline{0}$ yields an $\varepsilon - \text{DE}_{\hat{\alpha}(q)}$ with $\varepsilon > 0$ related to the mesh size of the underlying triangulation. Therefore, such a point q will be called a $\text{DE}_{\hat{\alpha}(q)}$ of $\hat{Z}$ or an approximate $\text{DE}_{\hat{\alpha}(q)}$. The short-term adjustment process operates in the n-dimensional subset $\underline{C}^{n+1}$. To obtain the path generated by it one can apply the product-ray algorithm described in Doup and Talman (1987) or the exponent-ray algorithm given in Doup et al. (1987) to the function $\hat{Z}$ on the set $\underline{C}^{n+1}$. Due to the special properties of the function $\hat{z}$ and therefore of $\hat{Z}$ we will be able to derive several interesting properties of the path. After the short-term adjustment process has reached a DE_0 of $\hat{Z}$ in $\underline{C}^{n+1}$, it generates a path of approximate $\text{DE}_{\hat{\alpha}(q)}$s in $\hat{C}^{n+1}$. This part of the price and quantity adjustment process does not correspond to any simplicial algorithm considered before in the literature. Next, a simplicial algorithm will be described that generates the path followed by both the short-term and the long-term price and quantity adjustment process.

To describe the algorithm, let an initial state v in the relative interior of $\underline{C}^{n+1}$ be given. Notice that for any such point it holds that $\hat{l}_j(v) < 0$, $\forall j \in I_n$, and $\hat{L}_j(v) > 0$, $\forall j \in I_n$. Therefore, there is neither complete supply rationing nor complete demand rationing, and in general no component of $\hat{z}(v)$ will be equal to zero. The algorithm now proceeds by increasing any component $j \in I_n$ of v for which $\hat{z}_j(v) > 0$ and decreasing any component $j \in I_n$ of v for which $\hat{z}_j(v) < 0$. To formalize this, let S be the set of n-dimensional sign vectors given by

$$S = \left\{ s \in \mathbb{R}^n | s_j \in \{ -1, 0, +1 \}, \forall j \in I_n \right\}.$$

For any $s \in S$, define the sets $I^-(s)$, $I^0(s)$, and $I^+(s)$ by $I^-(s) = \{ j \in I_n | s_j = -1 \}$, $I^0(s) = \{ j \in I_n | s_j = 0 \}$, and $I^+(s) = \{ j \in I_n | s_j = +1 \}$, so $I^-(s)$, $I^0(s)$, and $I^+(s)$ are the sets of negative, zero, and positive components of s, respectively. Let $i^-(s)$, $i^0(s)$, and $i^+(s)$ denote the number of components in these respective sets. For any $s \in S \setminus \{\underline{0}\}$, define the $i^0(s)$-dimensional set $\underline{C}^{n+1}(s)$ by

$$\underline{C}^{n+1}(s) = \left\{ q \in \underline{C}^{n+1} | q_j = 0, \forall j \in I^-(s), \text{ and } q_j = 1, \forall j \in I^+(s) \right\}$$

and define the $(i^0(s) + 1)$-dimensional set $A(s)$ by

$$A(s) = \text{co}\left(\{ v \} \cup \underline{C}^{n+1}(s) \right),$$

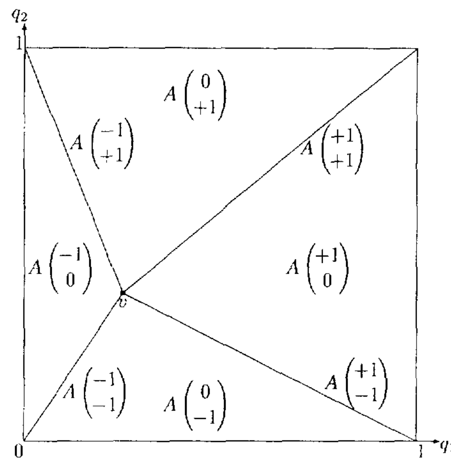

Fig. 1. The sets $A(s)$ for $n=2$.

where co denotes the convex hull of a set. Moreover, the set $A(\underline{0})$ is defined by $A(\underline{0})=\hat{C}^{n+1}$. The sets $A(s)$, $s \neq \underline{0}$ are illustrated for $n=2$ in Fig. 1. Let the first n components of the functions $\hat{\bar{z}}$ and $\hat{\bar{Z}}$ be denoted by $\hat{z}$ and $\hat{Z}$, respectively. Observe that a point $q \in \hat{C}^{n+1}$ satisfies the properties of Eq. (6) for the function $\hat{\bar{Z}}$ if and only if $q \in A(s)$ and $\text{sgn}(\hat{Z}(q))=s$ for some $s \in S \setminus \{\underline{0}\}$. Moreover, a point $q \in \hat{C}^{n+1}$ satisfies property Eq. (7) for the function $\hat{\bar{Z}}$ if and only if $q \in A(s)$ and $\text{sgn}(\hat{Z}(q))=s$ for $s=\underline{0}$.

Now, let $G(\underline{0})$ be a triangulation in $(n+1)$-simplices of the $(n+1)$-dimensional set $A(\underline{0})$ such that for any $s \in S$ the restriction of $G(\underline{0})$ to $A(s)$ induces a triangulation $G(s)$ in $(i^0(s)+1)$-simplices of $A(s)$. Such a triangulation of $\hat{C}^{n+1}$ is said to be a proper triangulation. A proper triangulation of $\hat{C}^{n+1}$ with arbitrarily small chosen mesh size exists and can be obtained by adapting the so-called V-triangulation on a cube developed in Doup and Talman (1987). For the remainder of this section some proper triangulation of $\hat{C}^{n+1}$ is assumed to be given.

Let $s \in S$ be a sign vector with $i^0(s)=t$, for some $t \in I_n^0$, and let $\sigma(q^0, \ldots, q^{t+1})$ be a $(t+1)$-dimensional simplex of $G(s)$. Consider solutions $(\bar{\lambda}_0, \ldots, \bar{\lambda}_{t+1}, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)}) \in \mathbb{R}^{n+2}$ to the following system of $(n+1)$ equations:

$$\sum_{h \in I_{t+1}^0} \lambda_h \begin{pmatrix} \hat{z}(q^h) \\ \hline 1 \end{pmatrix} - \sum_{j \in I^-(s) \cup I^+(s)} \mu_j \begin{pmatrix} s_j e(j) \\ 0 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \hline 1 \end{pmatrix}, \tag{9}$$

where $e(j)$ denotes the jth n-dimensional unit vector. If $\bar{\lambda}_h \leq 0$, $\forall h \in I_{t+1}^0$, and $\bar{\mu}_j \geq 0$, $\forall j \in I^-(s) \cup I^+(s)$, then $(\bar{\lambda}_0, \ldots, \bar{\lambda}_{t+1}, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)})$ is called an admissible solution to Eq. (9).

Notice that if $\sigma(q^0, \ldots, q^{t+1}) \in G(s)$ for some $s \in S$ and $(\bar{\lambda}_0, \ldots, \bar{\lambda}_{t+1}, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)})$ is an admissible solution to Eq. (9), then the point q given by $q = \sum_{h \in I^0_{t+1}} \bar{\lambda}_h q^h$ is an element of σ satisfying Eq. (6) for $\hat{Z}$ if $s \neq 0$ and Eq. (7) for $\hat{Z}$ if $s = 0$.

An admissible solution $(\bar{\lambda}_0, \ldots, \bar{\lambda}_{t+1}, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)})$ to Eq. (9) such that two or more variables are equal to zero is called a degenerate admissible solution to Eq. (9). Following the literature on simplicial algorithms, we make the non-degeneracy condition that there are no degenerate admissible solutions. Notice that the non-degeneracy condition is a very weak assumption, since Eq. (9) is a system with $n+2$ unknowns and $n+1$ equations, whereas letting two unknowns being equal to zero yields two more equations. Moreover, the number of possible systems, i.e. all simplices in $\bigcup_{s \in S} G(s)$, is finite. Clearly, if the function $\hat{z}$ is such that the non-degeneracy condition is not satisfied, then this assumption will be fulfilled for an arbitrarily small perturbation of $\hat{z}$. Furthermore, it is possible to circumvent this non-degeneracy condition completely by using lexicographic pivoting, see Todd (1976). However, lexicographic pivoting implies a much higher mathematical complexity and therefore we simply make the non-degeneracy condition in this paper. It is important to point out that the non-degeneracy condition has some interesting economic implications in our model due to the fact that $\hat{z}$ satisfies the boundary behaviour of Lemma 2. It will guarantee that the price and quantity adjustment process will not reach states of the economy with complete supply rationing or complete demand rationing on some market, i.e. if q is generated by the price and quantity adjustment process, then $0 < q_j < 1$, $\forall j \in I_n$. Indeed, the following lemma makes clear that there are no admissible solutions on the boundary of $\hat{C}^{n+1}$ where $q_j = 0$ or $q_j = 1$ for some $j \in I_n$ if the non-degeneracy condition is satisfied.

Lemma 4. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \bar{p})$ *satisfy Assumptions* 1–3. *Let G be a proper triangulation of* $\hat{C}^{n+1}$ *and assume that the non-degeneracy condition is satisfied. Let a sign vector* $s \in S$, *a simplex* $\sigma(q^0, \ldots, q^{t+1}) \in G(s)$ *and an admissible solution* $(\bar{\lambda}_0, \ldots, \bar{\lambda}_{t+1}, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)})$ *to Eq. (9) be given. Let* $\bar{q} = \sum_{h \in I^0_{t+1}} \bar{\lambda}_h q^h$. *Then* $\bar{q}_j \in (0, 1)$, $\forall j \in I_n$.

Proof. Suppose the lemma is not true and there exists $k \in I_n$ such that $\bar{q}_k = 0$ or $\bar{q}_k = 1$. First, consider the case where $\bar{q}_k = 0$. Since v is a point of the relative interior of $\underline{C}^{n+1}$ it follows that $k \in I^-(s) \cup I^0(s)$ and that $\bar{q}$ lies in a facet of σ, so there is $h' \in I^0_{t+1}$ with $\bar{\lambda}_{h'} = 0$. Without loss of generality it can be assumed that $h' = t+1$. Therefore, by the non-degeneracy condition, $\bar{\lambda}_h > 0$, $\forall h \in I^0_t$, and $\bar{\mu}_j > 0$, $\forall j \in I^-(s) \cup I^+(s)$, and hence $q^h_k = 0$, $\forall h \in I^0_t$. Notice that $\sum_{h \in I^0_t} \bar{\lambda}_h \hat{\underline{z}}(q^h) = \sum_{j \in I^-(s) \cup I^+(s)} \bar{\mu}_j s_j e(j)$.

Suppose $k \in I^-(s)$. Then, using Lemma 2, $0 \leq \sum_{h \in I^0_t} \bar{\lambda}_h \hat{\underline{z}}_k(q^h) = -\bar{\mu}_k < 0$, a contradiction. Consequently, $k \in I^0(s)$.

Since $k \in I^0(s)$ it follows that $\sum_{h \in I_t^0} \bar{\lambda}_h \hat{\underline{z}}_k(q^h) = 0$. Since $q_k^h = 0$, $\forall h \in I_t^0$, it follows by Lemma 2 that $\hat{\underline{z}}_k(q^h) \geq 0$, $\forall h \in I_t^0$, and therefore $\hat{\underline{z}}_k(q^h) = 0$, $\forall h \in I_t^0$. Hence, row k of the matrix M given by

$$M = \left[\begin{pmatrix} \hat{\underline{z}}(q^h) \\ 1 \end{pmatrix}_{h \in I_t^0}, \begin{pmatrix} -s_j e(j) \\ 0 \end{pmatrix}_{j \in I^-(s) \cup I^+(s)} \right]$$

is the zero vector and the rank of M is at most n. Since the system $Mx = (0^\top, 1)^\top$ has a solution $(\bar{\lambda}_0, \ldots, \bar{\lambda}_t, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)})$ with $\hat{\lambda}_h > 0$, $\forall h \in I_t^0$, $\bar{\mu}_j > 0$, $\forall j \in I^-(s) \cup I^+(s)$, and since the rank of M is at most n, there is a ray of such solutions. Hence, there is a solution with $\bar{\lambda}_h \geq 0$, $\forall h \in I_t^0$, $\mu_j \geq 0$, $\forall j \in I^-(s) \cup I^+(s)$, whereas $\bar{\lambda}_h = 0$ for some $h \in I_t^0$ or $\bar{\mu}_j = 0$ for some $j \in I^-(s) \cup I^+(s)$, a contradiction to the non-degeneracy condition. The case where $\bar{q}_k = 1$ is completely symmetric. QED

For some $s \in S$ and a simplex $\sigma(q^0, \ldots, q^{t+1}) \in G(s)$, let $(\bar{\lambda}_0, \ldots, \bar{\lambda}_{t+1}, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)})$ be an admissible solution to Eq. (9) with $\bar{\lambda}_{h'} = 0$ for some $h' \in I_{t+1}^0$. Then the point $\bar{q} = \sum_{h \in I_{t+1}^0 \setminus \{h'\}} \bar{\lambda}_h q^h$ is a point satisfying the properties of the adjustment process belonging to the facet $\tau(q^0, \ldots, q^{h'-1}, q^{h'+1}, \ldots, q^{t+1})$ of σ opposite the vertex $q^{h'}$. Such a facet is called an s-complete (or complete) facet. More precisely, a facet $\tau(q^0, \ldots, q^t)$ of a simplex $\sigma \in G(s)$ is s-complete for some sign vector $s \in S$ with $i^0(s) = t$ if the system of equations

$$\sum_{h \in I_t^0} \lambda_h \begin{pmatrix} \hat{\underline{z}}(q^h) \\ 1 \end{pmatrix} - \sum_{j \in I^-(s) \cup I^+(s)} \mu_j \begin{pmatrix} s_j e(j) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

has a solution $(\bar{\lambda}_1, \ldots, \bar{\lambda}_t, (\bar{\mu}_j)_{j \in I^-(s) \cup I^+(s)})$ satisfying $\bar{\lambda}_h \geq 0$, $\forall h \in I_t^0$, and $\bar{\mu}_j \geq 0$, $\forall j \in I^-(s) \cup I^+(s)$, called an admissible solution to Eq. (10). Notice that the non-degeneracy condition implies $\bar{\lambda}_h > 0$, $\forall h \in I_t^0$, and $\bar{\mu}_j > 0$, $\forall j \in I^-(s) \cup I^+(s)$. Two facets τ and $\bar{\tau}$ are said to be adjacent complete facets if

1. $\tau \neq \bar{\tau}$ and for some s, both τ and $\bar{\tau}$ are s-complete facets of the same simplex $\sigma \in G(s)$, or
2. $\tau \neq \hat{\tau}$, for some $s \neq \underline{0}$, τ is an s-complete facet of $\bar{\tau}$ and there exists some $j \in I^-(s) \cup I^+(s)$ such that $\bar{\tau}$ is an $\bar{s}$-complete simplex in $G(s)$ with $\bar{s}_j = 0$ and $\bar{s}_k = s_k$ for all $k \neq j$, or
3. $\tau = \bar{\tau}$ and for some $s \neq \underline{0}$, $j \in I^-(s) \cup I^+(s)$, $k \in I^0(s)$ we have that τ is both an s-complete facet of a simplex $\sigma \in G(s)$ and $\bar{\tau}$ is an $\bar{s}$-complete facet of a simplex $\bar{\sigma} \in G(\bar{s})$ with $\bar{s}_j = 0$, $\bar{s}_k \in \{-1, +1\}$ and $\bar{s}_h = s_h$ for all $h \neq j, k$.

Consider the simplex $\{v\}$. This 0-simplex can only be an s-complete facet for some $s \in S$ if $i^0(s) = 0$. It follows easily that $\{v\}$ is an s-complete facet of the unique simplex $\sigma(v, q^1) \in G(s)$ containing $\{v\}$ as a facet, where $s = \text{sgn}(\underline{z}(v))$ and the admissible solution to the system in Eq. (10) corresponding to s and $\{v\}$ is given by $\lambda_0 = 1$ and $\mu_j = |\hat{\underline{z}}_j(v)|$, $\forall j \in I_n$. The admissible solution corresponding to the system in Eq. (9) for the simplex $\sigma(v, q^1)$ is given by $\lambda_0 = 1$, $\lambda_1 = 0$, and $\mu_j = |\hat{z}_j(v)|$, $\forall j \in I_n$. Notice that the non-degeneracy condition implies $\mu_j > 0$, $\forall j \in \bar{I}_n$. Hence, it follows immediately that there is no other sign vector $s \in S$ for which $\{v\}$ is s-complete.

The following lemma shows that there is exactly one adjacent complete facet to $\{v\}$. Moreover, if τ is an s-complete facet belonging to $\bar{C}^{n+1}$, (hence τ is a $\underline{0}$-complete facet), then there is also exactly one adjacent complete facet to τ. Finally, if τ is an s-complete facet for some $s \in S$, and τ is not equal to $\{v\}$ and does not belong to $\bar{C}^{n+1}$, then there are exactly two adjacent complete facets to τ.

Lemma 5. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ satisfy Assumptions 1–3. Let G be a proper triangulation of $\hat{C}^{n+1}$ and assume that the non-degeneracy condition is satisfied. Let a sign vector $s \in S$ and an s-complete facet $\tau(q^0, \ldots, q^t)$ of a simplex in $G(s)$ be given. If $\tau = \{v\}$ or if $\tau \subset \bar{C}^{n+1}$, then there is exactly one adjacent complete simplex to τ. Otherwise, there are exactly two adjacent complete simplices to τ.

Proof. Let $\tau = \{v\}$. Then τ is s-complete for a uniquely determined sign vector $s \in S$. Since τ lies in the relative boundary of $A(s)$ there is a unique simplex $\sigma(v, q^1) \in G(s)$ having τ as a facet. Make a linear programming pivot step with the vector $(\hat{z}(q^1)^\top, 1)^\top$ in the system Eq. (10) corresponding to s and τ. By the non-degeneracy condition exactly one of the variables λ_0, μ_j, $j \in I_n$, becomes zero. If λ_0 becomes zero, then $\{q^1\}$ is an s-complete facet of a simplex in $G(s)$. If, say, $\mu_{j'} = 0$, then $\sigma(v, q^1)$ is an $\bar{s}$-complete facet of a uniquely determined simplex in $G(\bar{s})$, where $\bar{s}_{j'} = 0$ and $\bar{s}_j = s_j$, $\forall j \in I_n \setminus \{j'\}$. This yields exactly one adjacent complete simplex to τ. It is clear that there cannot be any other.

Let $\tau(q^0, \ldots, q^t)$ be an s-complete facet in $\bar{C}^{n+1}$. Then, τ is $\underline{0}$-complete and $t = n$. Since τ lies in the relative boundary of $A(\underline{0})$ there is exactly one simplex, say $\sigma(q^0, \ldots, q^{n+1})$, in $G(\underline{0})$ having τ as a facet. Make a linear programming pivot step with the vector $(\hat{z}(q^{n+1})^\top, 1)^\top$ in the system Eq. (10) corresponding to $s = \underline{0}$ and τ. By the non-degeneracy condition exactly one of the variables λ_h, $h \in \bar{I}_n^0$, becomes zero, say $\lambda_{h'}$. Therefore, the facet of σ opposite the vertex $\{q^{h'}\}$ is a $\underline{0}$-complete facet of a simplex in $G(\underline{0})$. This yields exactly one adjacent complete simplex to τ. It is clear that there cannot be any other adjacent complete simplex.

Let $\tau(q^0, \ldots, q^t)$ be an s-complete facet such that neither $\tau = \{v\}$ nor τ is a subset of $\bar{C}^{n+1}$. There are two possibilities: either τ lies in the relative boundary of $A(s)$ or τ lies in the relative interior of $A(s)$. Suppose τ lies in the relative boundary of $A(s)$. Then there is a unique simplex $\sigma(q^0, \ldots, q^{t+1})$ in $G(s)$ containing τ as a facet. Make a linear programming pivot step with the vector $(\hat{z}(q^{t+1})^\top, 1)^\top$ in the system Eq. (10) corresponding to s and τ. By the non-degeneracy condition exactly one of the variables λ_h, $h \in I_t^0$, μ_j, $j \in I^-(s) \cup I^+(s)$, becomes zero. If $\lambda_{h'}$ becomes zero, then the facet of σ opposite $q^{h'}$ is an

s-complete facet of a simplex in $G(s)$. If $\mu_{j'} = 0$, then σ is an $\bar{s}$-complete facet of a uniquely determined simplex in $G(\bar{s})$, where $\bar{s}_{j'} = 0$ and $\bar{s}_j = s_j$, $\forall j \in I_n \setminus \{j'\}$. This yields exactly one adjacent complete simplex to τ. Since τ lies in the relative boundary of $A(s)$, but τ is neither a subset of $\bar{C}^{n+1}$ nor of the boundary of $A(s)$ where $q_j = 0$ or $q_j = 1$ for some $j \in I_n$ by Lemma 4, it holds that τ is a simplex of $G(\hat{s})$ for a unique sign vector $\hat{s} \in S$ with $i^0(\hat{s}) = t - 1$. Let $j' \in I^0(s)$ be the unique component such that $\hat{s}_{j'} \neq 0$. Make a linear programming pivot step with the vector $(s_{j'} e(j')^\top, 0)^\top$ in the system Eq. (10) corresponding to s and τ. By the non-degeneracy condition exactly one of the variables λ_h, $h \in I_t^0$, μ_j, $j \in I^-(s) \cup I^+(s)$, becomes zero. If $\lambda_{h'}$ becomes zero, then the facet of τ opposite $q^{h'}$ is an $\hat{s}$-complete facet of τ. If $\mu_{\hat{j}} = 0$, then τ is an $\tilde{S}$-complete facet of a uniquely determined simplex $\sigma(q^0, \ldots, q^t, \bar{q}^{t+1})$ in $G(\tilde{s})$, where $\tilde{s}_{\hat{j}} = 0$ and $\tilde{s}_j = \hat{s}_j$, $\forall j \in I_n \setminus \{\hat{j}\}$. In this case τ is both an s-complete facet of a simplex in $G(s)$ and an $\tilde{s}$-complete facet of a simplex in $G(\tilde{s})$, with $s \neq \tilde{s}$, and is therefore adjacent complete to itself. It is clear that there cannot be any other.

Consider the case with τ lying in the relative interior of $A(s)$. Then there are exactly two simplices, say $\sigma(q^0, \ldots, q^{t+1})$ and $\sigma(q^0, \ldots, q^t, \bar{q}^{t+1})$, containing τ as a facet. Make a linear programming pivot step with the vector $(\bar{z}(q^{t+1})^\top, 1)^\top$ and with the vector $(\bar{z}(\bar{q}^{t+1})^\top, 1)^\top$, respectively, in the system Eq. (10) corresponding to s and τ. Using the same arguments as before, this yields exactly two adjacent complete simplices to τ. It is clear that there cannot be any other. QED

Lemma 6. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \bar{p})$ *satisfy Assumptions 1–3. Let G be a proper triangulation of* $\hat{C}^{n+1}$ *and assume that the non-degeneracy condition is satisfied. Then there is a unique finite sequence of different pairs of sign vectors and simplices* $(s^1, \tau^1), \ldots, (s^M, \tau^M)$ *such that* $\tau^1 = \{v\}$, $\tau^M \subset \bar{C}^{n+1}$, τ^k *is an* s^k*-complete facet, and any two successive simplices in the sequence are adjacent complete simplices.*

Proof. Let $\tau^1 = \{v\}$, which is $\mathrm{sgn}(\bar{z}(v))$-complete. Let τ^2, with corresponding sign vector s^2, be the unique adjacent complete simplex to τ^1, which exists according to Lemma 5. Now, for $k \in \mathbb{N} \setminus \{1\}$, it holds that either $\tau^k \subset \bar{C}^{n+1}$ and there exists no adjacent complete simplex other than τ^{k-1} by Lemma 5, or there exists by Lemma 5 a unique simplex τ^{k+1}, with corresponding sign vector s^{k+1}, that is not equal to τ^{k-1}. Proceeding this way, there exists a finite number M such that either $\tau^M \subset \bar{C}^{n+1}$ or, by the finiteness of the number of pairs (s, τ) such that τ is an s-complete facet of a simplex in $G(s)$, the pair (s^M, τ^M) has been generated before. However, from the door-in-door-out principle of Lemke and Howson (1964) it follows that each pair (s, τ), representing an s-complete facet τ of a simplex in $G(s)$ for some $s \in S$, can be generated at most once. Hence, $\tau^M \subset \bar{C}^{n+1}$. QED

The algorithm generating the sequence of adjacent complete simplices as described in Lemma 6 is given below. In the description of the algorithm, $\tau^k = \text{co}(\{q^h | h \in I_t^0\})$ will denote an s^k-complete facet currently generated by the procedure, where s^k with $i^0(s^k) = t$ is a sign vector generated by the procedure and induces the set $A(s^k)$ and the triangulation $G(s^k)$ of $A(s^k)$ in which the procedure generates simplices. The algorithm operates as follows.

Algorithm

Step 0. Set $t = 0$ and $k = 1$. Set $\tau^k = \{v\}$, $s^k = \text{sgn}(\hat{z}(v))$, and let q^{t+1} be the unique point in $A(s^k)$ such that $\sigma^k(v, q^{t+1})$ is a $(t+1)$-simplex of $G(s^k)$. Go to Step 1.

Step 1. Let σ be equal to the convex hull of $\tau^k \cup \{q^{t+1}\}$. Make a linear programming pivot step with the vector $(\hat{z}(q^{t+1})^\top, 1)^\top$ in the system Eq. (10) corresponding to s^k and τ^k. Then exactly one of the variables λ_h, $h \in I_t^0$, μ_j, $j \in I^-(s^k) \cup (I^+(s^k)$, becomes equal to zero. If for some $h' \in I_t^0$, $\lambda_{h'} = 0$, then set $s^{k+1} = s^k$ and go to Step 2. If for some $j' \in I^-(s^k) \cup I^+(s^k)$, $\mu_{j'} = 0$, then set $\bar{s} = s^k$ and go to Step 3.

Step 2. Increase the value of k by 1 and let τ^k be the facet of σ opposite $q^{h'}$. If $\tau^k \subset \bar{C}^{n+1}$, then the algorithm terminates. If $\tau^k \in G(\bar{s})$ for some $\bar{s} \in S$, then go to Step 4. Otherwise, there is exactly one $(t+1)$-simplex $\bar{\sigma} \in G(s^k)$ such that $\bar{\sigma} \neq \sigma$ and τ^k is a facet of $\bar{\sigma}$. Go to Step 1 with q^{t+1} as the unique vertex of $\bar{\sigma}$ opposite τ^k.

Step 3. Define s^{k+1} by $s_{j'}^{k+1} = 0$ and $s_j^{k+1} = \bar{s}_j$, $\forall j \in I_n \setminus \{j'\}$. There is a unique simplex $\bar{\sigma} \in G(s^{k+1})$ having σ as a facet. Increase the values of k and t by 1 and go to Step 1 with q^{t-1} as the unique vertex of $\bar{\sigma}$ opposite σ and $\tau^k = \sigma$.

Step 4. Let σ be equal to τ^k. Make a linear programming pivot step with the vector $(\bar{s}_j e(j)^\top, 0)^\top$ in the system Eq. (10) corresponding to s^k and τ^k, where $j \in I_n$ is such that $s_j^k = 0$ and $\bar{s}_j \neq 0$. Then exactly one of the variables λ_h, $h \in I_t^0$, μ_j, $j \in I^-(s^k) \cup I^+(s^k)$, becomes equal to zero. If for some $h' \in I_t^0$, $\lambda_{h'} = 0$, then set $s^{k+1} = \bar{s}$, decrease the value of t by 1 and go to Step 2. If for some $j' \in I^-(s^k) \cup I^+(s^k)$, $\mu_{j'} = 0$, then decrease the value of t by 1 and go to Step 3.

Let the assumptions of Lemma 6 be satisfied and let $(s^1, \tau^1), \ldots, (s^M, \tau^M)$ be all different pairs of sign vectors and simplices successively generated by the algorithm. For every pair (s^k, τ^k), $k \in I_M$, it holds that τ^k is an s^k-complete facet of an $(i^0(s^k)+1)$-simplex of $G(s^k)$. For every $k \in I_M$, define $t^k = i^0(s^k)$, let $\tau^k = \tau^k(q^0, \ldots, q^{t^k})$ and define $\bar{q}^k \in \tau^k$ by

$$\bar{q}^k = \sum_{h \in I_{t^k}^0} \bar{\lambda}_h q^h$$

with $\bar{\lambda}_h$ following from the admissible solution to Eq. (10) corresponding to s^k

and τ^k. For $t \in \mathbb{R}$, define $\lfloor t \rfloor$ as the smallest integer which is less than or equal to t. Finally, define the piecewise linear, continuous function $\pi:[0, 1] \to \hat{C}^{n+1}$ by

$$\pi(t) = (1-(M-1)t + \lfloor (M-1)t \rfloor)\bar{q}^{1+\lfloor (M-1)t \rfloor} + ((M-1)t - \lfloor (M-1)t \rfloor)\bar{q}^{2+\lfloor (M-1)t \rfloor}, \forall t \in [0,1),$$
$$\pi(1) = \bar{q}^M.$$

Clearly, π generates a piecewise linear path in $\hat{C}^{n+1}$ connecting the point $q^1 = \{v\}$ with a point $q^M \in \overline{C}^{n+1}$. Furthermore, every point $q \in \pi([0, 1]) \cap \underline{C}^{n+1}$ satisfies Eq. (6) for the piecewise linear approximation $\hat{Z}$, and every point $q \in \pi([0, 1]) \cap (\hat{C}^{n+1} \setminus \underline{C}^{n+1})$ satisfies Eq. (7) for $\hat{Z}$, as is proved in the following theorem.

Theorem 1. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ satisfy Assumptions 1–3. Let G be a proper triangulation of $\hat{C}^{n+1}$ and assume that the non-degeneracy condition is satisfied. Then, for every $q \in \pi([0, 1]) \cap \underline{C}^{n+1}$ there exists a real number $\gamma \in [0, 1]$ such that, for every $j \in I_n$,

$$\begin{aligned} q_j &= \gamma v_j && \text{if } \hat{Z}_j(q) < 0 \\ \gamma q_j \le q_j &\le 1 - \gamma(1 - v_j) && \text{if } \hat{Z}_j(q) = 0 \\ q_j &= 1 - \gamma(1 - v_j) && \text{if } \hat{Z}_j(q) > 0. \end{aligned}$$

Furthermore, for all $q \in \pi([0, 1]) \cap (\hat{C}^{n+1} \setminus \underline{C}^{n+1})$ it holds that $\underline{\hat{Z}}(q) = 0$.

Proof. Consider the pairs (s^k, τ^k) and (s^{k+1}, τ^{k+1}) for some $k \in I_{M-1}$. Let $\sigma^k = \text{co}(\tau^k \cup \tau^{k+1})$ and let $\bar{s}^k$ be such that $I^-(\bar{s}^k) = I^-(s^k) \cup I^-(s^{k+1})$, $I^0(\bar{s}^k) = I^0(s^k) \cap I^0(s^{k+1})$, and $I^+(\bar{s}^k) = I^+(s^k) \cup I^+(s^{k+1})$. For $\bar{q}^k$ and $\bar{q}^{k+1}$, induced by the admissible solutions to Eq. (10) corresponding to (s^k, τ^k) respectively (s^{k+1}, τ^{k+1}), it holds that they are also induced by admissible solutions y^k, $y^{k+1} \in \mathbb{R}^{n+2}$ to Eq. (9) corresponding to $(\bar{s}^k, \sigma^k)$. When y^k, y^{k+1} are admissible solutions, then so is $\lambda y^k + (1-\lambda) y^{k+1}$, $\forall \lambda \in [0, 1]$. Then, since $\sigma^k \in G(\bar{s}^k)$ for all $k \in I_{M-1}$, it follows that for every $q \in \pi([0, 1]) \cap \underline{C}^{n+1}$, and thus $q \in G(s)$ for some $s \neq \underline{0}$, Eq. (6) is satisfied for $\hat{Z}$, whereas for every $q \in \pi([0, 1]) \cap (\hat{C}^{n+1} \setminus \underline{C}^{n+1}$, and thus $q \in G(\underline{0})$, Eq. (7) is satisfied for $\underline{\hat{Z}}$. QED

6. Accuracy analysis

In Section 5 we have proved that the price and quantity adjustment process, applied to an arbitrary piecewise linear approximation $\hat{Z}$ of $\hat{z}$ satisfying a non-degeneracy condition, yields the existence of a piecewise linear path of points in $\hat{C}^{n+1}$ that connects the starting point $\{v\}$ with a zero point of $\hat{Z}$ in $\overline{C}^{n+1}$. For every point along the path that belongs to $\underline{C}^{n+1}$, Eq. (6) holds for the piecewise linear approximation, whereas for every point along the path that belongs to $\hat{C}^{n+1} \setminus \underline{C}^{n+1}$, Eq. (7) holds for the approximation $\underline{\hat{Z}}$.

In this section it will be shown that this path follows the price and quantity adjustment process related to $\hat{z}$ arbitrarily close when the mesh-size of the triangulation is taken small enough. To this end, observe that the properties, as given by Eq. (6), of points $q \in \underline{C}^{n+1} \setminus \{v\}$ that are reached by the price and quantity adjustment process are equivalent to

$$\begin{aligned} &\hat{z}_j(q) \le 0 && \text{if } q_j = \gamma v_j \\ &\hat{z}_j(q) = 0 && \text{if } \gamma v_j < q_j < 1 - \gamma(1 - v_j) \\ &\hat{z}_j(q) \ge 0 && \text{if } q_j = 1 - \gamma(1 - v_j). \end{aligned}$$

For $q = v$ the properties in Eq. (6) are trivially satisfied by taking $\gamma = 1$, both for $\hat{z}$ and for $\hat{Z}$. The following result shows that the other points related to the adjustment process for $\hat{Z}$ approximately satisfy the properties Eqs. (6) and (7) for $\hat{z}$.

Theorem 2. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ satisfy Assumptions 1–3. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every proper triangulation G of $\hat{C}^{n+1}$ satisfying mesh$(G) < \delta$ *and for which the non-degeneracy condition is satisfied, it holds that*

1. *for every $q \in \pi((0, 1]) \cap \underline{C}^{n+1}$, for every $j \in I_n$,*

$$\begin{aligned} &\hat{z}_j(q) < \varepsilon && \text{if } q_j = \gamma v_j \\ -\varepsilon < &\hat{z}_j(q) < \varepsilon && \text{if } \gamma v_j < q_j < 1 - \gamma(1 - v_j) \\ &\hat{z}_j(q) > -\varepsilon && \text{if } q_j = 1 - \gamma(1 - v_j). \end{aligned}$$

2. *for every $q \in \pi([0, 1]) \cap (\hat{C}^{n+1} \setminus \underline{C}^{n+1})$,*

$$\|\hat{z}(q)\|_\infty < \varepsilon.$$

Proof. Given $\varepsilon > 0$, define $\varepsilon' = (1 - \bar{\beta})(\sum_{j \in I_n} \tilde{p}_j)^{-1}\varepsilon$ and let $\bar{\varepsilon} = \min\{\varepsilon, \varepsilon'\}$. Since $\hat{C}^{n+1}$ is compact and $\hat{z}$ is continuous, there exists $\delta > 0$ such that q^1, $q^2 \in \hat{C}^{n+1}$, and $\|q^1 - q^2\|_\infty < \delta$ implies $\|\hat{z}(q^1) - \hat{z}(q^2)\|_\infty < \bar{\varepsilon}$. Let G be a proper triangulation of $\hat{C}^{n+1}$ satisfying mesh$(G) < \delta$ and being such that the non-degeneracy condition is satisfied. Furthermore, for all $q \in \hat{C}^{n+1}$ there exists $\lambda \in \mathbb{R}_+^{n+1}$ such that $\sum_{j \in I_{n+1}} \lambda_j = 1$ and $q = \sum_{j \in I_{n+1}} \lambda_j q^j$, with q^j the vertices of an n-simplex $\sigma(q^1, \ldots, q^{n+1})$ of G containing q. Consequently, for all $q \in \hat{C}^{n+1}$,

$$\|\hat{Z}(q) - \hat{z}(q)\|_\infty = \|\sum_{j \in I_{n+1}} \lambda_j(\hat{z}(q^j) - \hat{z}(q))\|_\infty < \bar{\varepsilon}$$

and therefore, for all $j \in I_{n+1}$,

$$\hat{Z}_j(q) - \bar{\varepsilon} < \hat{z}_j(q) < \hat{Z}_j(q) + \bar{\varepsilon}. \tag{11}$$

Now, for every $q \in \pi((0, 1]) \cap \underline{C}^{n+1}$ there exists $\mu_j \in \mathbb{R}_+$, $\forall j \in I_n$, and $\gamma \in [0, 1]$, such that for every $j \in I_n$,

$$\hat{Z}_j(q) = -\mu_j \quad \text{if } q_j = \gamma v_j$$

$$\hat{Z}_j(q) = 0 \quad \text{if } \gamma v_j < q_j < 1 - \gamma(1 - v_j)$$

$$\hat{Z}_j(q) = \mu_j \quad \text{if } q_j = 1 - \gamma(1 - v_j).$$

Combining this with Eq. (11) yields Part 1 of the theorem. For $q \in \pi([0, 1]) \cap (\hat{C}^{n+1} \setminus \underline{C}^{n+1})$ it holds that $\hat{Z}_j(q) = 0$, $\forall j \in I_n$, and thus, using Eq. (11), $|\hat{z}_j(q)| < \bar{\varepsilon}$. Consequently, $|\Sigma_{j \in I_n} \hat{p}_j(q)\hat{z}_j(q)| \leq \Sigma_{j \in I_n} |\hat{p}_j(q)\hat{z}_j(q)| < (1 - \underline{\beta})^{-1} \Sigma_{j \in I_n} \tilde{p}_j \bar{\varepsilon} \leq \varepsilon$. But then, since $\hat{p}(q) \cdot \hat{z}(q) = 0$ by Lemma 1, we have that $|\hat{z}_{n+1}(q)| = |\Sigma_{j \in I_n} \hat{p}_j(q)\hat{z}_j(q)| < \varepsilon$. QED

Notice that Theorem 2 implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every proper triangulation G of $\hat{C}^{n+1}$ with mesh$(G) < \delta$ and for which the non-degeneracy condition is satisfied it holds that every point q in $\hat{C}^{n+1} \setminus \underline{C}^{n+1}$ generated by the algorithm is an $\varepsilon - \mathrm{DE}_{\hat{\alpha}(q)}$ and that every point q in $\underline{C}^{n+1}$ generated by the algorithm is an $\varepsilon -$ Walrasian equilibrium. This follows easily once it is shown that demand rationing on the market of commodity $j \in I_n$ cannot be binding for such a point q if $q_j \leq 2/3$. Without loss of generality assume that $\varepsilon < \min_{i \in I_m, j \in I_n}(\tilde{p} \cdot w/\tilde{p}_j - w_j + w_j^i)$. Then, for every $i \in I_m$, for every $j \in I_n$, it follows that

$$\hat{d}_j^i(q) - w_j^i \leq \hat{z}_j(q) + w_j - w_j^i < \varepsilon + w_j - w_j^i < \frac{\tilde{p} \cdot w}{\tilde{p}_j} = \hat{L}_j(q) \text{ if } q_j \leq \tfrac{2}{3}.$$

The next result shows that the path of the approximate adjustment process gets arbitrarily close to the set Q_v. This does not exclude that the set Q_v contains points which are never approximated by the adjustment process related to $\hat{Z}$. For instance, in case the set Q_v displays a bifurcation, it is possible that the approximate adjustment process converges to one of the branches. In case the adjustment process related to $\hat{z}$ is convergent, also the converse follows, i.e. every point of Q_v gets arbitrarily close to the path of the approximate adjustment process. It follows that if the adjustment process related to $\hat{z}$ is convergent, then any sequence of paths related to the approximate adjustment process with mesh size going to zero converges to Q_v. For a non-empty compact set S of $\mathbb{R}^k$ define the distance function $g_S: \mathbb{R}^k \to \mathbb{R}$ by

$$g_S(\bar{s}) = \min_{s \in S} \|s - \bar{s}\|_2, \ \forall \bar{s} \in \mathbb{R}^k$$

Notice that both the sets Q_v and $\pi([0, 1])$ are compact.

Theorem 3. Let the economy $\mathscr{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{p})$ *satisfy Assumptions* 1–3. *Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every proper triangulation* G *of* $\hat{C}^{n+1}$ *satisfying* mesh$(G) < \delta$ *and for which the non-degeneracy condition is* *satisfied, it holds that, for every* $t \in [0, 1]$, $g_{Q_\iota}(\pi(t)) < \varepsilon$. *If the price and quantity adjustment process is convergent then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every proper triangulation G of* $\hat{C}^{n+1}$ *satisfying* mesh$(G) < \delta$ *and for which the non-degeneracy condition is satisfied, it holds that, for every* $q \in Q_\iota$, $g_{\pi([0,1])}(q) < \varepsilon$.

Proof. Suppose the first part of the theorem is not true. Then there exists $\varepsilon > 0$ such that for every $r \in \mathbb{N}$ there exists a proper triangulation G^r of $\hat{C}^{n+1}$ satisfying mesh$(G^r) < 1/r$, for which the non-degeneracy condition is satisfied, and there exists $t^r \in [0, 1]$ such that $g_{Q_\iota}(\pi^r(t^r)) \geq \varepsilon$. By Hildenbrand (1974, Proposition 1, p. 16), the sequence $\{\pi^r([0, 1])\}_{r \in \mathbb{N}}$ has a convergent subsequence which we also denote by $\{\pi^r([0, 1])\}_{r \in \mathbb{N}}$. Using the continuity of $\hat{z}$ and Theorem 2 it follows easily that the closed limit of this sequence is a subset of Q. By Mas-Colell (1985, Theorem A.5.1.(ii), p. 10), the closed limit is connected since every $\pi^r([0, 1])$ is connected. Therefore, the closed limit is a subset of Q_ι. Consider any accumulation point of the sequence $\{\pi^r(t^r)\}_{r \in \mathbb{N}}$, say $\bar{q}$, so $\bar{q} \in Q_\iota$. So,

$$0 = g_{Q_\iota}(\bar{q}) \geq \inf\left\{g_{Q_\iota}(\pi^r(t^r)) | r \in \mathbb{N}\right\} \geq \varepsilon,$$

a contradiction.

Suppose the second part of the theorem is not true. Then there exists $\varepsilon > 0$ such that for every $r \in \mathbb{N}$ there exists a proper triangulation G^r of $\hat{C}^{n+1}$ satisfying mesh$(G) < \delta$ for which the non-degeneracy condition is satisfied, and there exists $q^r \in Q_\iota$ such that $g_{\pi^r([0,1])}(q^r) \geq \varepsilon$. Without loss of generality, the sequence $\{q^r\}_{r \in \mathbb{N}}$ is convergent, say to the element $\bar{q}$. Also without loss of generality, by Hildenbrand (1974, Proposition 1, p. 16), the sequence $\{\pi^r([0, 1])\}_{r \in \mathbb{N}}$ is convergent, say to the set Π. By the first part of the theorem it follows that $\Pi \subset Q_\iota$. Clearly, $\bar{q} \notin \Pi$. Since the adjustment process Q_ι is convergent there exists a homeomorphism f:$[0, 1] \to Q_\iota$, where $f(0) = v$ and $f(1) = q^*$ with q^* inducing a Walrasian equilibrium of $\mathscr{E}$ and $q^*_{n+1} = \bar{\beta}$. Using that Q_ι is an arc and that there exists a neighbourhood N of q^* such that $q \in N$ and $\hat{z}(q) = \underline{0}$ implies that q induces a Walrasian equilibrium of $\mathscr{E}$, it follows that there is no $q \in Q_\iota$ such that $q \neq q^*$ and $q_{n+1} = \bar{\beta}$. Hence, $\pi^r(0) \to f(0) = v$ and $\pi^r(1) \to f(1) = q^*$. Moreover, there is $\bar{t} \in (0, 1)$ such that $\bar{q} = f(\bar{t})$. By Mas-Colell (1985, Theorem A.5.1.(ii), p. 10), Π is connected. However, $\Pi \subset f([0, 1] \setminus \{\bar{t}\})$, $f(0) \in \Pi$, $f(1) \in \Pi$ and f is a homeomorphism, so Π is not connected, a contradiction. QED

References

Bénassy, J.-P., 1993. Nonclearing markets: microeconomic concepts and macroeconomic applications. Journal of Economic Literature 31, 732–761.

Böhm, V., 1993. Recurrence in Keynesian macroeconomic models. In: Gori, F., Geronazzo, L., Galeotti, M. (Eds.), Nonlinear Dynamics in Economics and the Social Sciences. Springer-Verlag, Berlin, pp. 69–94.

Day, R.H., Pianigiani, G., 1991. Statistical dynamics and economics. Journal of Economic Behaviour and Organization 16, 37–83.
Doup, T.M., van den Elzen, A.H., Talman, A.J.J., 1987. Simplicial algorithms for solving the nonlinear complementarity problem on the simplotope. In: Talman, A.J.J., van der Laan, G. (Eds.), The Computation and Modelling of Economic Equilibria. North-Holland, Amsterdam, pp. 125–154.
Doup, T.M., Talman, A.J.J., 1987. A new simplicial variable dimension algorithm to find equilibria on the product space of unit simplices. Mathematical Programming 37, 319–355.
Drèze, J.H., 1975. Existence of an exchange economy under price rigidities. International Economic Review 16, 310–320.
Garcia, C.B., Zangwill, W.I., 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall Series in Computational Mathematics, Prentice-Hall, Englewood Cliffs.
Gordon, R.J., 1990. What is new-Keynesian economics? Journal of Economic Literature 28, 1115–1171.
Herings, P.J.J., 1997. A globally and universally stable price adjustment process. Journal of Mathematical Economics 27, 163–193.
Herings, P.J.J., 1994. A globally and universally stable quantity adjustment process for an economy with price rigidities. CentER Discussion Paper 94111, Tilburg University, Tilburg.
Herings, P.J.J., van der Laan, G., Talman, A.J.J., Venniker, R., 1997. Equilibrium adjustment of disequilibrium prices. Journal of Mathematical Economics 27, 53–77.
Hildenbrand, W., 1974. Core and Equilibria of a Large Economy. Princeton University Press, Princeton.
Kamiya, K., 1990. A globally stable price adjustment process. Econometrica 58, 1481–1485.
van der Laan, G., 1980. Equilibrium under rigid prices with compensation for the consumers. International Economic Review 21, 63–73.
van der Laan, G., Talman, A.J.J., 1987a. Adjustment processes for finding economic equilibria. In: Talman, A.J.J., van der Laan, G. (Eds.), Computation and Modelling of Economic Equilibria. North-Holland, Amsterdam, pp. 85–123.
van der Laan, G., Talman, A.J.J., 1987b. A convergent price adjustment process. Economics Letters 23, 119–123.
Laroque, G., 1981. A comment on "Stable spillovers among substitutes". Review of Economic Studies 48, 355–361.
Lemke, C.E., Howson, J.T., 1964. Equilibrium points of bimatrix games. SIAM Journal of Applied Mathematics 12, 413–423.
Mankiw, N.G., 1994. Macroeconomics, 2nd ed. Worth Publishers, New York.
Mas-Colell, A., 1985. The Theory of General Economic Equilibrium, A Differentiable Approach. Cambridge University Press, Cambridge.
Movshovich, S.M., 1994. A price adjustment process in a rationed economy. Journal of Mathematical Economics 23, 305–321.
Saari, D.G., Simon, C., 1978. Effective price mechanisms. Econometrica 46, 1097–1125.
Scarf, H., 1960. Some examples of global instability of the competitive equilibrium. International Economic Review 1, 157–172.
Scarf, H., 1973. The Computation of Economic Equilibria. Yale University Press, New Haven.
Smale, S., 1976. A convergent process of price adjustment and global Newton methods. Journal of Mathematical Economics 3, 107–120.
Todd, M.J., 1976. The Computation of Fixed Points and Applications. Lecture Notes in Economics and Mathematical Systems 124, Springer-Verlag, Berlin.
Veendorp, E.C.H., 1975. Stable spillovers among substitutes. Review of Economic Studies 42, 445–456.