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THE COMPUTATION OF A CONTINUUM OF CONSTRAINED EQUILIBRIA

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The equilibria of a model of an exchange economy with price rigidities are called constrained equilibria. A simplicial algorithm on the unit cube is proposed to compute an approximation of a continuum of zero points of the excess demand correspondence of such an economy. This continuum of zero points corresponds to a continuum of constrained equilibria of the economy, linking the two so-called trivial equilibria. Moreover, the convergence properties of the algorithm and the accuracy of the approximate constrained equilibria obtained by the algorithm are discussed. Finally, an example is given as an illustration.

1. Introduction. Over the last two decades many important and basic existence problems raised in economic theory and game theory have been successfully solved by constructive approaches. Most of the literature on this issue derives from the pioneering work of Scarf (1967); see also Scarf (1973). Scarf introduced an algorithm based on the relationship between Sperner's lemma and Brouwer's fixed point theorem to generate a sequence of adjacent primitive sets in the unit simplex which terminates with an approximate fixed point or equilibrium price vector of an Arrow-Debreu economic model within a finite number of iterations. Kuhn (1968) proposed such an algorithm with simplices instead of primitive sets. Later more efficient and sophisticated algorithms were developed, e.g., by Eaves (1972), Merrill (1972), and van der Laan and Talman (1979). In these algorithms a sequence of adjacent simplices of a triangulation is generated in order to approximate a fixed point of a continuous function.

There is a clear interaction between the developments in economic theory and game theory on one side and those in fixed point or zero point algorithms on the other side. Shoven and Whalley (1992) and Kaneko and Yamamoto (1986) use simplicial algorithms to compute an approximate equilibrium in a general equilibrium model with taxation and an economic model with indivisible commodities, respectively. By providing constructive equilibrium existence proofs the computational approach might give additional insight into the problems under consideration. An example of the influence of game theory on simplicial fixed point or zero point algorithms is given in the paper of van der Laan and Talman (1982). Whereas the algorithms developed earlier were suitable for a problem only defined on the unit simplex or on $\mathbb{R}^n$, in their paper an algorithm on the Cartesian product of unit simplices was developed, which is more natural when computing Nash equilibria of noncooperative games.

In some recent work the problem of computing equilibria when the conditions of Kakutani's fixed point theorem are not satisfied and computing refinements of equilibria has been addressed. Brown, DeMarzo and Eaves (1993) give a procedure to compute equilibria in a model with incomplete markets. In this model the equilibrium existence problem can be formulated as a fixed point problem for some function, but

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either the domain of this function is not convex, or the function is not continuous. Using the special structure of the discontinuities they are still able to compute an equilibrium for this model in generic cases. Yamamoto (1993) and Talman and Yang (1994) consider the problem of computing a proper Nash equilibrium in a noncooperative game.

In this paper we consider an exchange economy with price rigidities and quantity rationing as introduced by Drèze (1975). This model is often studied in economic literature. In such an economy the situation in which under rationing demand equals supply is called a constrained equilibrium. It is well known that there exist two trivial constrained equilibria, one under complete demand rationing and the other under complete supply rationing. We propose a simplicial zero point algorithm on the unit cube with vector labelling, which starts from one of the trivial constrained equilibria, generates a piecewise linear path contained in a sequence of adjacent simplices of varying dimension, and terminates with the other trivial constrained equilibrium within a finite number of iterations. We demonstrate that every point on the path generated by the algorithm yields an approximate constrained equilibrium. The algorithm differs from other simplicial algorithms in the sense that a whole set of approximate constrained equilibria is generated. By using a limit argument we are able to prove that there exists a connected set of constrained equilibria linking both trivial constrained equilibria. Therefore, the method allows us to find all kinds of constrained equilibria. Moreover, the excess demand is allowed to be a correspondence instead of a function. To do this, we have to deal with specific degeneracy problems. For some work to compute one specific kind of constrained equilibria induced by continuous excess demand functions, we refer to van der Laan (1982) and Cornielje and van der Laan (1986).

The rest of the paper is organized as follows. In §2 we introduce the model of an exchange economy with price rigidities and rationing. In §3 we present the steps of the algorithm in detail and prove the convergence of the algorithm. In §4 we analyze the accuracy of the approximate solutions obtained by the algorithm and show that a connected set of constrained equilibria containing the two trivial constrained equilibria is approximated. Finally, an illustration of the algorithm can be found in §5.

2. A model of an economy with price rigidities. In the following, for $k \in \mathbb{N}$, I_k denotes the set of integers $\{1, \ldots, k\}$, C^k denotes the k-dimensional unit cube, so $C^k = \{q \in \mathbb{R}^k | \forall j \in I_k,\ 0 \le q_j \le 1\}$, 0^k (1^k) denotes the vector in $\mathbb{R}^k$ for which each component is equal to 0 (1), and E^k denotes the $k \times k$ identity matrix. If $x, y \in \mathbb{R}^k$, then $x \ge y$ means $x_j \ge y_j$, $\forall j \in I_k$, $x > y$ means $x \ge y$ and $\exists j \in I_k$ such that $x_j > y_j$, and $x \gg y$ means $x_j > y_j$, $\forall j \in I_k$. The set $\{x \in \mathbb{R}^k | x \gg 0\}$ is denoted by $\mathbb{R}^k_{++}$. If S is a subset of $\mathbb{R}^k$, then $\text{Int}(S)$ denotes the interior of S in $\mathbb{R}^k$.

In this section a model of an exchange economy with price rigidities and a rationing system is presented and the equilibrium existence problem for such an economy is formulated as a zero point problem. An exchange economy with price rigidities is defined by $E = (\{X^i, \succcurlyeq^i, w^i\}_{i=1}^m, P)$. There are m consumers indexed by $i = 1, \ldots, m$ and n commodities indexed by $j = 1, \ldots, n$. For $i = 1, \ldots, m$, consumer i is characterized by a consumption set X^i, a preference ordering $\succcurlyeq^i$ on X^i, and a vector of initial endowments w^i. The set of admissible prices is denoted by P. Since the set of admissible prices is allowed to be restricted, a Walrasian equilibrium price system at which demand equals supply for every commodity does not necessarily exist. In case of excess demands or excess supplies, rationing can be used to obtain a situation where the excess demands on all markets are zero. A rationing scheme gives lower and upper bounds on the excess demands of all consumers. The description of the economy is extended by a rationing system. This rationing system specifies the feasible rationing schemes in the economy and is given by functions $\tilde{l}$ and $\tilde{L}$, following Weddepohl (1987). Both $\tilde{l}$ and $\tilde{L}$ have mn components and describe rationing schemes for each of the consumers on their excess supplies and on their excess demands, respectively, being permitted in the economy. For $i = 1, \dots, m$ and $j = 1, \dots, n$, component $(i-1)n + j$ of $\tilde{l}$ is denoted by $\tilde{l}^i_j$ and component $(i-1)n + j$ of $\tilde{L}$ is denoted by $\tilde{L}^i_j$. With respect to the economy E and the rationing system $(\tilde{l}, \tilde{L})$ the following assumptions are made:

(A1) For every $i \in I_m$, X^i is a convex, closed, nonempty subset of $\mathbb{R}^n_+$ and $X^i + \mathbb{R}^n_+ \subset X^i$.

(A2) For every $i \in I_m$, the preference ordering $\succcurlyeq^i$ on X^i is transitive, complete, continuous, strongly monotonic, and convex.

(A3) For every $i \in I_m$, the initial endowments w^i are an element of $\mathrm{Int}(X^i)$.

(A4) The set of admissible prices is equal to

$$P = \left\{ p \in \mathbb{R}^n_+ | \underline{p}_j \le p_j \le \bar{p}_j, \forall j \in I_n \right\},$$

for given $\underline{p}, \bar{p} \in \mathbb{R}^n_{++}$ where $\underline{p}_j \le \bar{p}_j$, $\forall j \in I_n$.

(A5) The functions $\tilde{l}\colon C^n \to -\mathbb{R}^{mn}_+$ and $\tilde{L}\colon C^n \to \mathbb{R}^{mn}_+$ specifying the rationing system are continuous on C^n and satisfy for every $i \in I_m$, $j \in I_n$, and $q^1, q^2 \in C^n$,

$$\tilde{l}^i_j(q^1) = \tilde{l}^i_j(r^1), \quad \text{if } r^1 \in C^n \text{ and } q^1_j = r^1_j, \qquad \tilde{L}^i_j(q^2) = \tilde{L}^i_j(r^2), \quad \text{if } r^2 \in C^n \text{ and } q^2_j = r^2_j,$$

$$\tilde{l}^i_j(q^1) = 0, \quad \text{if } q^1_j = 0, \qquad \tilde{L}^i_j(q^2) = 0, \quad \text{if } q^2_j = 0,$$

$$\tilde{l}^i_j(q^1) < -w^i_j, \quad \text{if } q^1_j = 1, \qquad \tilde{L}^i_j(q^2) > \sum_{h \ne i} w^h_j, \quad \text{if } q^2_j = 1.$$

As has been shown in Debreu (1959), Assumptions (A1) and (A2) imply that it is possible to represent the preferences of consumer $i \in I_m$ by a continuous utility function u^i from X^i into $\mathbb{R}$. Assumption (A4) implies that the admissible prices are restricted to lie between a lower bound and an upper bound.

The constrained budget set of consumer $i \in I_m$ at price $p \in P$ and his individual rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$, is denoted by $B^i(p, l^i, L^i)$ and is given by

$$B^i(p, l^i, L^i) = \{ x^i \in X^i | p \cdot x^i \le p \cdot w^i, l^i \le x^i - w^i \le L^i \}.$$

The rationing scheme $(l, L) = ((l^1, \dots, l^m), (L^1, \dots, L^m)) \in -\mathbb{R}^{mn}_+ \times \mathbb{R}^{mn}_+$ is permitted in the economy if there exists $q^1, q^2 \in C^n$ such that $(l, L) = (\tilde{l}(q^1), \tilde{L}(q^2))$. Assumption (A5) guarantees that the rationing scheme on the market of commodity $j \in I_n$ is completely determined by component j of q^1 and q^2. The value q^1_j determines the amount of supply rationing on the market of commodity j and the value of q^2_j determines the amount of demand rationing on the market of commodity j. When $q^1_j = 0$ there is complete supply rationing on the market of commodity j and if $q^1_j = 1$ then there is no binding supply rationing on this market. When $q^2_j = 0$ there is complete demand rationing on the market of commodity j and if $q^2_j = 1$ then there will be no binding demand rationing on this market in an equilibrium of the economy. As is shown by Herings (1992), Assumption (A5) admits many possible specifications of the rationing system, for example uniform rationing, rationing determined by a priority system, etc.

The demand of consumer $i \in I_m$ at price $p \in P$ and his individual rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$ is denoted by $\delta^i(p, l^i, L^i)$ and is given by

$$\delta^i(p, l^i, L^i) = \left\{ x^i \in B^i(p, l^i, L^i) | u^i(x^i) = \max_{y^i \in B^i(p, l^i, L^i)} u^i(y^i) \right\}.$$

The following definition of a constrained equilibrium is similar to the one given in Drèze (1975).

DEFINITION 2.1 (CONSTRAINED EQUILIBRIUM). A constrained equilibrium of the economy $E = (\{X^i, \succcurlyeq^i, w^i\}_{i=1}^m, P)$ with rationing system $(\tilde{l}, \tilde{L})$ is an element $(x^{*1}, \ldots, x^{*m}, l^*, L^*, p^*)$ of the set $\Pi_{i=1}^m X^i \times -\mathbb{R}_+^{mn} \times \mathbb{R}_+^{mn} \times P$ such that

(1) $\forall i \in I_m$: $x^{*i} \in \delta^i(p^*, l^{*i}, L^{*i})$;

(2) $\Sigma_{i=1}^m x^{*i} - \Sigma_{i=1}^m w^i = 0^n$;

(3) $\forall j \in I_n$: $x_j^{*h} - w_j^h = l_j^{*h}$ for some $h \in I_m$ implies $x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$, and $x_j^{*h} - w_j^h = L_j^{*h}$ for some $h \in I_m$ implies $x_j^{*i} - w_j^i > l_j^{*i}$, $\forall i \in I_m$;

(4) $\forall j \in I_n$: $p_j^* < \bar{p}_j$ implies $L_j^{*i} > x_j^{*i} - w_j^i$, $\forall i \in I_m$, and $p_j^* > \underline{p}_j$ implies $l_j^{*i} < x_j^{*i} - w_j^i$, $\forall i \in I_m$;

(5) $\exists q^{*1}, q^{*2} \in C^n$ such that $(l^*, L^*) = (\tilde{l}(q^{*1}), \tilde{L}(q^{*2}))$.

Consider the state of the market of some commodity $j \in I_n$ in a constrained equilibrium $(x^{*1}, \ldots, x^{*m}, l^*, L^*, p^*)$ of the economy E. Let $q^{*1}, q^{*2} \in C^n$ be such that $l^* = \tilde{l}(q^{*1})$ and $L^* = \tilde{L}(q^{*2})$. By Conditions 3 and 4 of Definition 2.1 there are three mutually exclusive possibilities on the market of commodity j. First, it may happen that for some consumer the supply rationing is binding on this market, i.e., there exists a consumer $h \in I_m$ such that $x_j^{*h} - w_j^h = l_j^{*h}$, so $0 \leq q_j^{*1} < 1$. Then, by Condition 3 of Definition 2.1, $x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$, so there is no binding demand rationing on the market of commodity j. Therefore, there is no loss of generality in assuming that $q_j^{*2} = 1$. Moreover, by Condition 4 of Definition 2.1, supply rationing can only be binding when the price is on its lower bound, i.e., $p_j^* = \underline{p}_j$. So, in this case the state of the market of commodity j is completely determined by the value of q_j^{*1}. The second possibility is that $l_j^{*i} < x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$, so there is no binding rationing on the market of commodity j. There is no loss of generality in assuming that $q_j^{*1} = 1$ and $q_j^{*2} = 1$. Clearly, the price of commodity j is between $\underline{p}_j$ and $\bar{p}_j$, so $\underline{p}_j \leq p_j^* \leq \bar{p}_j$. The state of the market of commodity j is completely determined by the value of p_j^*. Finally, the third possibility is that there exists a consumer $h \in I_m$ such that $x_j^{*h} - w_j^h = L_j^{*h}$, so $0 \leq q_j^{*2} < 1$. Then, by Condition 3 of Definition 2.1, $x_j^{*i} - w_j^i > l_j^{*i}$, $\forall i \in I_m$, so there is no binding supply rationing on the market of commodity j. There is no loss of generality in assuming that $q_j^{*1} = 1$. Moreover, by Condition 4 of Definition 2.1, $p_j^* = \bar{p}_j$, so the state of the market of commodity j is completely determined by the value of q_j^{*2}.

Let some commodity $j \in I_n$ be given. Motivated by the remarks in the previous paragraph, any of the three possible regimes on the market of commodity j will be described by one parameter $q_j \in [0,1]$. If $0 \leq q_j < \frac{1}{3}$, then the first possibility described above will occur with price $p_j = \underline{p}_j$ and rationing scheme $(l_j^i, L_j^i) = (\tilde{l}_j^i(q^1), \tilde{L}_j^i(q^2))$, $\forall i \in I_m$, where $q_j^1 = 3q_j$ and $q_j^2 = 1$. If $\frac{1}{3} \leq q_j \leq \frac{2}{3}$, then the second possibility will result with price $p_j = (2 - 3q_j)\underline{p}_j + (3q_j - 1)\bar{p}_j$ and rationing scheme $(l_j^i, L_j^i) = (\tilde{l}_j^i(q^1), \tilde{L}_j^i(q^2))$, $\forall i \in I_m$, where $q_j^1 = 1$ and $q_j^2 = 1$. If $\frac{2}{3} < q_j \leq 1$, then the third possibility will occur with price $p_j = \bar{p}_j$ and rationing scheme $(l_j^i, L_j^i) = (\tilde{l}_j^i(q^1), \tilde{L}_j^i(q^2))$, $\forall i \in I_m$, where $q_j^1 = 1$ and $q_j^2 = 3 - 3q_j$. Therefore, the price and rationing function $(\hat{p}, \hat{l}, \hat{L})$: $C^n \to P \times -\mathbb{R}_+^{mn} \times \mathbb{R}_+^{mn}$ is defined by

$$\hat{p}_j(q) = \max\left\{\underline{p}_j, \min\left\{(2 - 3q_j)\underline{p}_j + (3q_j - 1)\bar{p}_j, \bar{p}_j\right\}\right\}, \qquad \forall q \in C^n, \forall j \in I_n,$$

$$\hat{l}_j^i(q) = \tilde{l}_j^i(\min\{1^n, 3q\}), \qquad \forall q \in C^n, \forall i \in I_m, \forall j \in I_n,$$

$$\hat{L}_j^i(q) = \tilde{L}_j^i(\min\{1^n, 3\,1^n - 3q\}), \qquad \forall q \in C^n, \forall i \in I_m, \forall j \in I_n,$$

where the minimum of two vectors is taken componentwise. It can easily be shown that if only prices and rationing schemes are considered which are in the image set of the function $(\hat{p}, \hat{l}, \hat{L})$: $C^n \to P \times -\mathbb{R}^{mn}_{+} \times \mathbb{R}^{mn}_{+}$, then all constrained equilibria satisfying Definition 2.1 are obtained. Define the excess demand correspondence ζ: $C^n \to \mathbb{R}^n$ by

$$\zeta(q) = \sum_{i=1}^{m} \delta^i(\hat{p}(q), \hat{l}^i(q), \hat{L}^i(q)) - \sum_{i=1}^{m} \{w^i\}, \qquad \forall q \in C^m.$$

Then it holds that all zero points of ζ correspond to constrained equilibria and vice versa. More precisely, if $0^n \in \zeta(q^*)$ for some $q^* \in C^n$, then there exists $x^{*i} \in \delta^i(\hat{p}(q^*), \hat{l}^i(q^*), \hat{L}^i(q^*))$, $\forall i \in I_m$, such that $(x^{*1}, \dots, x^{*m}, \hat{l}(q^*), \hat{L}(q^*), \hat{p}(q^*))$ is a constrained equilibrium. Using the results in Herings (1992) the following theorem can be shown.

THEOREM 2.2. *Let the economy* $E = (\{X^i, \succcurlyeq^i, w^i\}_{i=1}^m, P)$ *with rationing system* $(\hat{l}, \hat{L})$ *be given and let Assumptions* (A1), (A2), (A3), (A4), *and* (A5) *be satisfied. Then the correspondence* ζ: $C^n \to \mathbb{R}^n$ *satisfies the following conditions*:

(1) ζ *is a nonempty valued, convex valued, upper semi-continuous correspondence such that* $\bigcup_{q \in C^n} \zeta(q)$ *is bounded*;

(2) $\forall q \in C^n$, $\forall z \in \zeta(q)$, $\forall j \in I_n$, $z_j \geq 0$ *if* $q_j = 0$;

(3) $\forall q \in C^n$, $\forall z \in \zeta(q)$, $\forall j \in I_n$, $z_j \leq 0$ *if* $q_j = 1$;

(4) $\forall q \in C^n$, $\forall z \in \zeta(q)$, $\hat{p}(q) \cdot z = 0$.

Using Conditions 2 and 4 it is not difficult to show that $q = 0^n$ corresponds to a constrained equilibrium where all the excess supply is rationed. From Conditions 3 and 4 it is easily seen that $q = 1^n$ corresponds to a constrained equilibrium where all the excess demand is rationed. In the following section an algorithm is presented which yields a piecewise linear path of points in C^n with end points 0^n and 1^n such that every point on the path corresponds to an approximate constrained equilibrium. This result will be used to show that there exists a connected set C in C^n containing 0^n and 1^n and satisfying $0^n \in \zeta(q)$, for any $q \in C$, so each point $q \in C$ corresponds to a constrained equilibrium.

3. The algorithm. In this section we introduce a simplicial variable dimension algorithm which will be used to approximate and prove the existence of a continuum of constrained equilibria. More generally, the algorithm can be used to compute a set of zero points for any correspondence ζ: $C^n \to \mathbb{R}^n$ satisfying the following condition.

CONDITION B. The correspondence ζ: $C^n \to \mathbb{R}^n$ satisfies:

(1) ζ is a nonempty valued, convex valued, upper semi-continuous correspondence such that $\bigcup_{q \in C^n} \zeta(q)$ is bounded;

(2) For every $q \in C^n$, there exists $z \in \zeta(q)$ such that, for every $j \in I_n$, $q_j = 0$ implies $z_j \geq 0$, and $q_j = 1$ implies $z_j \leq 0$;

(3) For every $q \in C^n$, for every $z \in \zeta(q)$, there exists $p \in \mathbb{R}^n_{++}$ such that $p \cdot z = 0$.

Notice that, by Theorem 2.2, the excess demand correspondence of an economy with price rigidities satisfies Condition B. More precisely, Condition B(1) is the same as Property 1 of Theorem 2.2, Condition B(2) is weaker than Properties 2 and 3 of Theorem 2.2, and Condition B(3) is weaker than Property 4 of Theorem 2.2.

In order to describe the algorithm, we need to introduce some notation first. A vector $s \in \mathbb{R}^n$ of which each component is equal to -1, 0, or $+1$ is called a sign

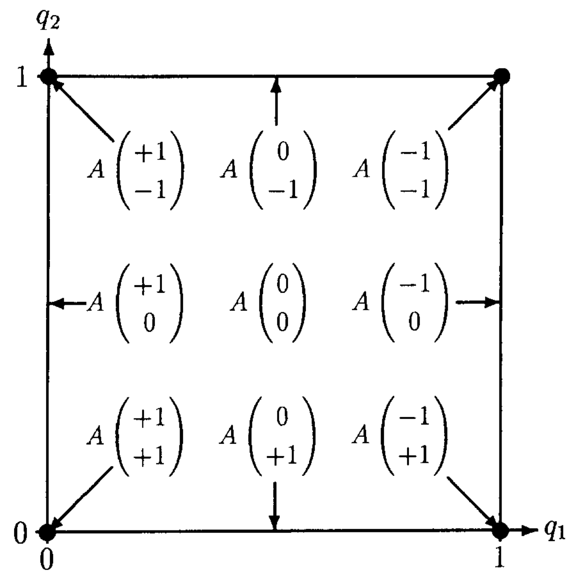

FIGURE 1.

vector. For a sign vector $s \in \mathbb{R}^n$ define the sets $I^-(s) = \{j \in I_n | s_j = -1\}$, $I^0(s) = \{j \in I_n | s_j = 0\}$, and $I^+(s) = \{j \in I_n | s_j = +1\}$. Moreover, the subset $A(s)$ of C^n is defined by

$$A(s) = \{x \in C^n | x_j = 0 \text{ if } s_j = +1, x_j = 1 \text{ if } s_j = -1\}.$$

It is easy to see that the dimension of $A(s)$ is equal to $|I^0(s)|$. Notice that the set $A(0^n)$ equals the set C^n. All 3^n possible sets $A(s)$ are illustrated in Figure 1 for $n = 2$.

For given $t \in \mathbb{N}$, a t-dimensional simplex or t-simplex, denoted by σ, is defined as the convex hull of $t + 1$ affinely independent points $x^1, \ldots, x^{t+1}$ of $\mathbb{R}^n$. We usually write $\sigma = \sigma(x^1, \ldots, x^{t+1})$ and call $x^1, \ldots, x^{t+1}$ the vertices of σ. A $(t-1)$-simplex being the convex hull of t vertices of a t-simplex σ is said to be a facet of σ. The facet $\tau(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{t+1})$ is called the facet of $\sigma(x^1, \ldots, x^{t+1})$ opposite to the vertex x^i. For k, $0 \leq k \leq t$, a k-simplex being the convex hull of $k + 1$ vertices of σ is said to be a k-face or face of σ. A finite collection $\mathscr{T}$ of n-simplices is a simplicial subdivision or triangulation of C^n if

(1) C^n is the union of all simplices in $\mathscr{T}$;

(2) The intersection of any two simplices in $\mathscr{T}$ is either the empty set or a common face of both.

It can be shown that for any sign vector s with $|I^0(s)| = t$ the collection of t-faces of the simplices in $\mathscr{T}$ lying in the set $A(s)$ induces a triangulation of $A(s)$. Since $\mathscr{T}$ is finite and C^n is compact, it can be shown that each facet τ of a t-simplex $\sigma \in A(s)$ either lies in the relative boundary of $A(s)$ and is only a facet of σ or is a facet of exactly one other t-simplex in $A(s)$. The mesh size of a triangulation $\mathscr{T}$ of C^n is defined by $\text{mesh}(\mathscr{T}) = \max\{\|x - y\|_\infty | x, y \in \sigma, \sigma \in \mathscr{T}\}$.

An example of a triangulation of C^n having an arbitrarily chosen mesh size and that can easily be implemented is the K-triangulation proposed in Freudenthal (1942). Let $r \in \mathbb{N}$ be given. The K-triangulation of C^n with mesh size r^{-1} is the collection of all n-simplices $\sigma_{(x^1, \pi)}$ with vertices $x^1, \ldots, x^{n+1}$ in C^n such that each component of x^1 is a multiple of r^{-1}, $\pi = (\pi_1, \ldots, \pi_n)$ is a permutation of elements

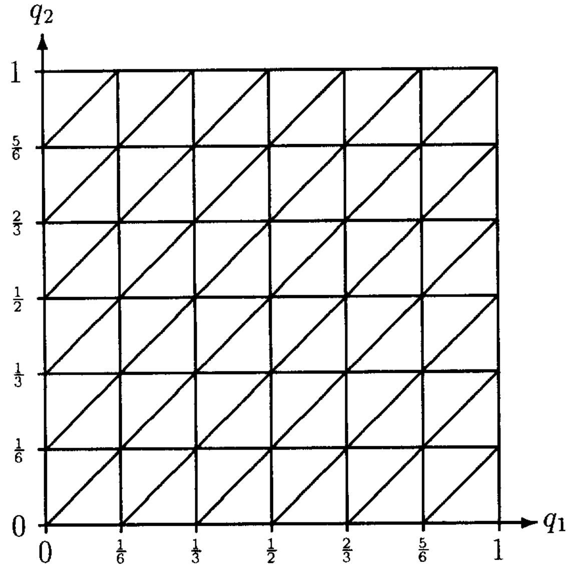

FIGURE 2.

of I_n, and for every $j \in I_n$, $x^{j+1} = x^j + r^{-1}e(\pi_j)$. Here, for $k \in I_n$, $e(k)$ denotes the kth n-dimensional unit vector. In the notation $e(k)$ the parameter n is omitted, since the dimension of $e(k)$ will always equal the number of commodities, n. For $n = 2$ and $r = 6$ the K-triangulation is illustrated in Figure 2.

Let ζ: $C^n \to \mathbb{R}^n$ be a correspondence satisfying Condition B. A function Z: $C^n \to \mathbb{R}^n$ is called a piecewise linear approximation of ζ with respect to a given triangulation $\mathcal{T}$ of C^n if for each vertex x of any $\sigma \in \mathcal{T}$ it holds that $Z(x) \in \zeta(x)$ and Z is affine on each simplex of $\mathcal{T}$. Moreover, in the sequel Z will be chosen such that the following boundary condition is satisfied.

CONDITION C. Let ζ: $C^n \to \mathbb{R}^n$ be a correspondence satisfying Condition B and let $\mathcal{T}$ be a triangulation of C^n. The piecewise linear approximation Z: $C^n \to \mathbb{R}^n$ of ζ with respect to $\mathcal{T}$ is such that for each vertex x in the triangulation $\mathcal{T}$ it holds that $Z_j(x) \geq 0$ if $x_j = 0$, and $Z_j(x) \leq 0$ if $x_j = 1$.

Notice that it is always possible to let a piecewise linear approximation Z of ζ with respect to a triangulation $\mathcal{T}$ of C^n satisfy Condition C if ζ satisfies Condition B.

To approximate and to prove the existence of a connected set of zero points of ζ lying in C^n and containing 0^n and 1^n, we constructively prove that for any piecewise linear approximation Z of ζ satisfying Condition C it holds that there exists a piecewise linear path f: $[0, 1] \to C^n$ of points in C^n connecting 0^n and 1^n, and having some specific properties. Let $\tilde{C}$ denote the image of $[0, 1]$ by f, so $\tilde{C} = f([0, 1])$. The piecewise linear path $\tilde{C}$ will be constructed such that for every point $q \in \tilde{C}$ there is a number $\beta \in \mathbb{R}$ satisfying

(1)
$$
\begin{aligned}
0 \leq Z_j(q) &\leq \beta \quad \text{if } q_j = 0,\\
Z_j(q) &= \beta \quad \text{if } 0 < q_j < 1,\\
0 \geq Z_j(q) &\geq \beta \quad \text{if } q_j = 1.
\end{aligned}
$$

As will be shown in the next section, if mesh($\mathcal{T}$) is small enough, then β will be arbitrarily close to zero for every $q \in \tilde{C}$, i.e., every point q on the path is an approximate zero point of ζ when the mesh size of the underlying triangulation $\mathcal{T}$ is taken small enough.

Let $\sigma(x^1, \ldots, x^{t+1})$ be a t-simplex in C^n and let s be a sign vector with $|I^0(s)| = t$. Let $i_1 < \cdots < i_{n-t}$ be such that $I^-(s) \cup I^+(s) = \{i_1, \ldots, i_{n-t}\}$. Consider solutions $(\lambda_1, \ldots, \lambda_{t+1}, \mu_1, \ldots, \mu_{n-t}, \beta) \in \mathbb{R}^{n+2}$ of the following system of equations:

$$\text{(2)} \qquad \sum_{j=1}^{t+1} \lambda_j \begin{pmatrix} 1 \\ Z(x^j) \end{pmatrix} + \sum_{j=1}^{n-t} \mu_j \begin{pmatrix} 0 \\ s_{i_j} e(i_j) \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1^n \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix}.$$

If $\lambda_j \geq 0$, $\forall j \in I_{t+1}$, and $\mu_j \geq 0$, $\forall j \in I_{n-t}$, then $(\lambda_1, \ldots, \lambda_{t+1}, \mu_1, \ldots, \mu_{n-t}, \beta)$ is called an admissible solution to (2). Notice that β is unrestricted and that $\sum_{j=1}^{t+1} \lambda_j = 1$. For an admissible solution to (2) corresponding to a simplex $\sigma(x^1, \ldots, x^{t+1})$ in $A(s)$ it holds that the point q satisfies (1), where $q = \sum_{j=1}^{t+1} \lambda_j x^j \in \sigma$. Notice that the linear system (2) has $n + 1$ equations and $n + 2$ variables and hence there is one degree of freedom. The piecewise linear path $\tilde{C}$ connecting 0^n and 1^n will be obtained by an algorithm that generates a sequence of adjacent simplices of varying dimension in $\mathcal{T}$ such that with respect to each simplex of this sequence there exists a set of admissible solutions. Each linear piece of the path $\tilde{C}$ within some simplex of $\mathcal{T}$ will be traced by making a pivot step in system (2) with one of the variables. Since β is an unconstrained variable, its corresponding column is not subject to leaving the system. No nondegeneracy assumptions are to be made. The pivot steps to be described below are inspired by Todd (1976) and Wright (1981), where algorithms are presented to compute a zero point of a correspondence $f: \mathbb{R}^n \to \mathbb{R}^n$. In their work lexicographic pivot rules are used in order to circumvent degeneracy.

An admissible solution to (2) is said to be degenerate if at least two of the variables λ_j, $j \in I_{t+1}$, and μ_j, $j \in I_{n-t}$, equal zero. For any piecewise linear approximation Z satisfying Condition C of a correspondence ζ satisfying Condition B with respect to any triangulation $\mathcal{T}$ of C^n it holds that $Z(0^n) = 0^n$ and $Z(1^n) = 0^n$. Hence $\lambda_1 = 1$, $\mu_j = 0$, $\forall j \in I_n$, and $\beta = 0$ is a degenerate admissible solution to (2) corresponding to the 0-simplex $\sigma(0^n)$ and any sign vector s with $|I^0(s)| = 0$. A similar completely degenerate admissible solution exists corresponding to the 0-simplex $\sigma(1^n)$. So, the usual nondegeneracy assumption that there exist no degenerate admissible solutions makes no sense for the problem under consideration.

A row vector $x \in \mathbb{R}^n$ is said to be lexicographically positive if it is nonzero and its first nonzero entry is positive. A matrix A is said to be semi-lexicopositive if each row except possibly the last row is lexicographically positive. For a $(t-1)$-simplex $\tau(x^1, \ldots, x^t)$ in C^n and a sign vector s with $|I^0(s)| = t$, the $(n+1) \times (n+1)$ matrix $A_{s,\tau}$ is defined by

$$A_{s,\tau} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ Z(x^1) & \cdots & Z(x^t) & s_{i_1} e(i_1) & \cdots & s_{i_{n-t}} e(i_{n-t}) & -1^n \end{pmatrix}.$$

DEFINITION 3.1. Let τ be a $(t-1)$-simplex of C^n and let s be a sign vector with $|I^0(s)| = t$. Then τ is s-complete if $A_{s,\tau}^{-1}$ exists and is semi-lexicopositive.

When τ is s-complete, then the first column of $A_{s,\tau}^{-1}$ corresponds to an admissible solution to (2) for a t-simplex σ being the convex hull of τ and some vertex $x^{t+1} \in C^n$, where $\lambda_{t+1} = 0$.

We will now describe an algorithm that generates a finite sequence of simplices starting with the simplex $\{0^n\}$ and terminating with the simplex $\{1^n\}$ satisfying that for every $(t-1)$-simplex τ in the sequence there exists a sign vector s with $|I^0(s)| = t$ such that τ is an s-complete simplex in $A(s)$. Moreover, any two successive simplices in the sequence either are facets of the same simplex or one is a facet of the other. We first show in Lemmas 3.2 and 3.3 that $\tau(0^n)$ and $\tau(1^n)$ are each s-complete 0-simplices in $A(s)$ with respect to some unique sign vector s with $|I^0(s)| = 1$. Lemmas 3.5 and 3.6 describe all possible situations that can occur when some s-complete simplex in $A(s)$ is obtained. Lemmas 3.2, 3.3, 3.5, and 3.6 are used in Theorem 3.7 to determine in a unique way the finite sequence of simplices described above. Then the detailed steps of the algorithm yielding this sequence are given and it is shown in Theorem 3.8 that the algorithm induces a piecewise linear path of points connecting 0^n and 1^n, having the properties given in (1).

LEMMA 3.2. *Let the sign vector $s \in \mathbb{R}^n$ be such that $s_j = +1$, $\forall j \in I_{n-1}$, and $s_n = 0$. Then the 0-simplex $\tau(0^n)$ is an s-complete simplex in $A(s)$ and is not an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for any other sign vector $\tilde{s} \in \mathbb{R}^n$.*

PROOF. Suppose that $\tau(0^n)$ is an $\tilde{s}$-complete simplex in $A(\tilde{s})$. Condition B(3) and Condition C imply that $Z(0^n) = 0^n$. Since τ is 0-dimensional and τ lies in $A(\tilde{s})$, it has to hold that $|I^0(\tilde{s})| = 1$ and $I^-(\tilde{s}) = \varnothing$. Then $A_{\tilde{s},\tau}$ is given by

$$A_{\tilde{s},\tau} = \begin{pmatrix} 1 & 0 & & 0 & 0 \\ & & \cdots & & \\ 0^n & e(i_1) & & e(i_{n-1}) & -1^n \end{pmatrix},$$

where $i_1 < i_2 < \cdots < i_{n-1}$. Define $i_0 = 0$ and $i_n = n + 1$. Let j be the unique element in the set $I^0(\tilde{s})$ and let $k \in I_n$ be such that $i_{k-1} < j < i_k$. Then

$$A_{\tilde{s},\tau}^{-1} = \begin{pmatrix} 1 & 0 & & 0 & 0 & 0 & & 0 \\ & & \cdots & & & & \cdots & \\ 0^n & e(i_1) & & e(i_{k-1}) & -1^n & e(i_k - 1) & & e(i_{n-1} - 1) \end{pmatrix}.$$

It is clear that this matrix is semi-lexicopositive if and only if $k = n$. Q.E.D.

LEMMA 3.3. *Let the sign vector $s \in \mathbb{R}^n$ be such that $s_1 = 0$ and $s_j = -1$, $\forall j \in I_n \setminus \{1\}$. Then the 0-simplex $\tau(1^n)$ is an s-complete simplex in $A(s)$ and is not an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for any other sign vector $\tilde{s}$.*

PROOF. Suppose that $\tau(1^n)$ is an $\tilde{s}$-complete simplex in $A(\tilde{s})$. Condition B(3) and Condition C imply that $Z(1^n) = 0^n$. Since τ is 0-dimensional and since τ lies in $A(\tilde{s})$, it holds that $|I^0(\tilde{s})| = 1$ and $I^+(\tilde{s}) = \varnothing$. Hence, $A_{\tilde{s},\tau}$ is given by

$$A_{\tilde{s},\tau} = \begin{pmatrix} 1 & 0 & & 0 & 0 \\ & & \cdots & & \\ 0^n & -e(i_1) & & -e(i_{n-1}) & -1^n \end{pmatrix},$$

where $i_1 < i_2 < \cdots < i_{n-1}$. Define $i_0 = 0$ and $i_n = n + 1$. Let j be the unique element in the set $I^0(\tilde{s})$ and let $k \in I_n$ be such that $i_{k-1} < j < i_k$. Then it is easily verified that

$$A_{\tilde{s},\tau}^{-1} = \begin{pmatrix} 1 & 0 & & 0 & 0 & 0 & & 0 \\ & & \cdots & & & & \cdots & \\ & & & & 1^{n-1} & & & \\ 0^n & -e(i_1) & & -e(i_{k-1}) & -1 & -e(i_k - 1) & & -e(i_{n-1} - 1) \end{pmatrix}.$$

Clearly, this matrix is semi-lexicopositive if and only if $k = 1$. Q.E.D.

The following lemma is well known in linear programming theory; see for example Murty (1983, p. 232). It will be very useful in proving Lemmas 3.5 and 3.6. Let A be a $k^1 \times k^2$ matrix. Then, for $j \in I_{k^1}$, $A_{j\cdot}$ denotes row j of the matrix A, and for $j \in I_{k^2}$, $A_{\cdot j}$ denotes column j of the matrix A.

LEMMA 3.4. *Let $A = (A_{\cdot 1}, \dots, A_{\cdot n+1})$ be a regular $(n+1) \times (n+1)$ matrix and let $x \in \mathbb{R}^{n+1}$. Let $k \in I_{n+1}$ and let $\tilde{A} = (A_{\cdot 1}, \dots, A_{\cdot k-1}, x, A_{\cdot k+1}, \dots, A_{\cdot n+1})$. Then either $(A^{-1}x)_k = 0$ and $\tilde{A}$ is singular, or $(A^{-1}x)_k \neq 0$ and*

$$\tilde{A}^{-1} = \begin{pmatrix} (A^{-1})_{1\cdot} - \dfrac{(A^{-1}x)_1}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ \vdots \\ (A^{-1})_{k-1\cdot} - \dfrac{(A^{-1}x)_{k-1}}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ \dfrac{1}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ (A^{-1})_{k+1\cdot} - \dfrac{(A^{-1}x)_{k+1}}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ \vdots \\ (A^{-1})_{n+1\cdot} - \dfrac{(A^{-1}x)_{n+1}}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \end{pmatrix}.$$

Lemma 3.4 is easily proved by showing that $\tilde{A}^{-1}\tilde{A} = E^{n+1}$. The next lemma describes the cases that may occur when a t-simplex σ in $A(s)$ has at least one s-complete facet τ and a lexicographic pivot step to $A_{s,\tau}$ is performed when the column in (2) to be added corresponds to the vertex of σ opposite τ.

LEMMA 3.5. *Let σ be a t-simplex of $A(s)$ where s is a sign vector with $|I^0(s)| = t$. If σ has an s-complete facet τ, then exactly one of the following cases holds:*

(1) The simplex σ is an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for precisely one sign vector $\tilde{s}$ with $|I^0(\tilde{s})| = t+1$ and no facets of σ other than τ are s-complete;

(2) The simplex σ has exactly one other s-complete facet $\tilde{\tau}$ and σ is not an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for any sign vector $\tilde{s}$ with $|I^0(\tilde{s})| = t+1$.

PROOF. Let x^{t+1} be the vertex of σ not contained in τ, and let $y = A_{s,\tau}^{-1}(1, Z(x^{t+1})^\top)^\top$. Since $A_{s,\tau}A_{s,\tau}^{-1} = E^{n+1}$, it holds that $\Sigma_{i=1}^{t}(A_{s,\tau}^{-1})_{i1} = 1$ and $\Sigma_{i=1}^{t}(A_{s,\tau}^{-1})_{ij} = 0$, $\forall j \in I_{n+1} \setminus \{1\}$. Suppose that the first n components of y are nonpositive. Then $0 \geq (1^{t^\top}, 0^{n-t+1^\top})y = (1^{t^\top}, 0^{n-t+1^\top})A_{s,\tau}^{-1}(1, Z(x^{t+1})^\top)^\top = (1, 0^{n^\top})(1, Z(x^{t+1})^\top)^\top = 1$. This is a contradiction. So, it is possible to choose $k \in I_n$ such that $(1/y_k)(A_{s,\tau}^{-1})_{k\cdot}$ is minimal according to the lexicographic ordering over all row vectors $(1/y_j)(A_{s,\tau}^{-1})_{j\cdot}$ for which $y_j > 0$ and $j \in I_n$. It is clear that k is uniquely determined, since otherwise $A_{s,\tau}^{-1}$ would be singular. Now it holds that either $k \in I_n \setminus I_t$ or $k \in I_t$.

If $k \in I_n \setminus I_t$, then let $\tilde{s}_{i_{k-t}} = 0$ and $\tilde{s}_j = s_j$ for $j \in I_n \setminus \{i_{k-t}\}$. Clearly, $\sigma \subset A(\tilde{s})$ and $|I^0(\tilde{s})| - 1$ equals the dimension of σ. The matrix $A_{\tilde{s},\sigma}$ is obtained by deleting column k of $A_{s,\tau}$ and adding the vector $(1, Z(x^{t+1})^\top)^\top$ between columns t and $t+1$. Using Lemma 3.4, $A_{\tilde{s},\sigma}^{-1}$ exists and is semi-lexicopositive. So, σ is an $\tilde{s}$-complete t-simplex in $A(\tilde{s})$.

If $k \in I_t$, then let $\tilde{\tau}$ be the facet of σ opposite the vertex x^k. Using Lemma 3.4 and the choice of k it follows that $A_{s,\tilde{\tau}}^{-1}$ exists and is semi-lexicopositive. Hence, $\tilde{\tau}$ is an s-complete simplex in $A(s)$.

It follows directly from Lemma 3.4 that if some other column is replaced, then the inverse of the new matrix is not semi-lexicopositive. Q.E.D.

The next lemma gives the cases that may occur when an s-complete $(t-1)$-simplex τ in $A(s)$ lies in a $(t-1)$-dimensional set $A(\tilde{s})$ and a lexicographic pivot step to $A_{s,\tau}$ is performed in the case the column in (2) to be added is equal to $(0, \tilde{s}_{\tilde{k}} e(\tilde{k})^\top)^\top$ for the unique element $\tilde{k}$ in the set $(I^-(\tilde{s}) \cup I^+(\tilde{s})) \setminus (I^-(s) \cup I^+(s))$.

LEMMA 3.6. *Let τ be an s-complete $(t-1)$-simplex of $A(s)$ where s is a sign vector with $|I^0(s)| = t$. Suppose that τ lies in $A(\tilde{s})$ where $\tilde{s}$ has $t-1$ zero components. Then exactly one of the following cases holds:*

(1) *The simplex τ is equal to $\{0^n\}$ or equal to $\{1^n\}$;*

(2) *The simplex τ is an $\bar{s}$-complete simplex in $A(\bar{s})$ for precisely one sign vector $\bar{s} \neq s$ with $|I^0(\bar{s})| = t$ and τ has no $\tilde{s}$-complete facets;*

(3) *Precisely one facet of τ is $\tilde{s}$-complete and τ is not an $\bar{s}$-complete simplex in $A(\bar{s})$ for any sign vector $\bar{s} \neq s$.*

PROOF. For some unique index $\tilde{k} \in I_n$, it must hold that $s_{\tilde{k}} = 0$ and $\tilde{s}_{\tilde{k}} \neq 0$. Let $y = A_{s,\tau}^{-1}(0, \tilde{s}_{\tilde{k}} e(\tilde{k})^\top)^\top$. Exactly one of the following three possibilities occurs: (i) $\tilde{s}_{\tilde{k}} = +1$ and $y_j \leq 0$, $\forall j \in I_n$; (ii) $\tilde{s}_{\tilde{k}} = -1$ and $y_j \leq 0$, $\forall j \in I_n$; (iii) $\exists j \in I_n$ such that $y_j > 0$.

If $\tilde{s}_{\tilde{k}} = +1$ and $y_j \leq 0$, $\forall j \in I_n$, then, since $A_{s,\tau} y = (0, \tilde{s}_{\tilde{k}} e(\tilde{k})^\top)^\top$, we have $y_1 = \cdots = y_t = 0$. So, $1 = \sum_{j=1}^{n+1} (A_{s,\tau})_{\tilde{k}+1,j} y_j = \sum_{j=t+1}^{n+1} (A_{s,\tau})_{\tilde{k}+1,j} y_j = -y_{n+1}$, where for the last equality it is used that $s_{\tilde{k}} = 0$. Hence, $y_{n+1} = -1$, and consequently

$$\sum_{j=t+1}^{n} (A_{s,\tau})_{\cdot j} y_j = \sum_{j=1}^{n-t} \begin{pmatrix} 0 \\ s_{i_j} e(i_j) \end{pmatrix} y_{t+j} = \begin{pmatrix} 0 \\ e(\tilde{k}) \end{pmatrix} - \begin{pmatrix} 0 \\ 1^n \end{pmatrix}.$$

Therefore, $t = 1$ and $s_{i_j} = +1$, $\forall j \in I_{n-t}$. So, $\tilde{s} = +1^n$ and therefore $\tau = \{0^n\}$. If $\tilde{s}_{\tilde{k}} = -1$ and $y_j \leq 0$, $\forall j \in I_n$, then again $y_1 = \cdots = y_t = 0$. So, $-1 = \sum_{j=t+1}^{n+1} (A_{s,\tau})_{\tilde{k}+1,j} y_j = -y_{n+1}$, and hence $y_{n+1} = 1$. Consequently,

$$\sum_{j=t+1}^{n} (A_{s,\tau})_{\cdot j} y_j = \sum_{j=1}^{n-t} \begin{pmatrix} 0 \\ s_{i_j} e(i_j) \end{pmatrix} y_{t+j} = \begin{pmatrix} 0 \\ 1^n \end{pmatrix} - \begin{pmatrix} 0 \\ e(\tilde{k}) \end{pmatrix}.$$

Therefore, $t = 1$ and $s_{i_j} = -1$, $\forall j \in I_{n-t}$. So, $\tilde{s} = -1^n$ and therefore $\tau = \{1^n\}$.

If there exists some $j \in I_n$ such that $y_j > 0$, then it is possible to choose $k \in I_n$ as in the proof of Lemma 3.5. Again, either $k \in I_n \setminus I_t$ or $k \in I_t$.

If $k \in I_n \setminus I_t$, then let $\bar{s}_{\tilde{k}} = \tilde{s}_{\tilde{k}}$, $\bar{s}_{i_{k-t}} = 0$, and $\bar{s}_j = \tilde{s}_j$, $\forall j \in I_n \setminus \{i_{k-t}, \tilde{k}\}$, and consider $A_{\bar{s},\tau}$. Using Lemma 3.4, the choice of k guarantees that $A_{\bar{s},\tau}^{-1}$ is semi-lexicopositive and therefore τ is $\bar{s}$-complete in $A(\bar{s})$.

If $k \in I_t$, let τ' be the facet of τ opposite to the vertex x^k. By Lemma 3.4 and the choice of k, $A_{\tilde{s},\tau'}^{-1}$ is semi-lexicopositive and hence τ' is $\tilde{s}$-complete in $A(\tilde{s})$.

It follows directly from Lemma 3.4 that if some other column of $A_{s,\tau}$ is replaced, then the inverse of the new matrix is not semi-lexicopositive. Q.E.D.

An s^1-complete simplex τ^1 in $A(s^1)$ and an s^2-complete simplex τ^2 in $A(s^2)$ are said to be adjacent complete simplices if $s^1 = s^2 = s$ and τ^1 and τ^2 are both facets of a simplex σ in $A(s)$, or if τ^1 is a facet of τ^2 and τ^2 is a simplex in $A(s^1)$, or if τ^2 is a facet of τ^1 and τ^1 is a simplex in $A(s^2)$. Using the lemmas above it can be shown that there exists a finite sequence of adjacent complete simplices of varying dimension connecting the simplices $\{0^n\}$ and $\{1^n\}$.

THEOREM 3.7. *Let $\mathcal{T}$ be a triangulation of C^n, let ζ: $C^n \to \mathbb{R}^n$ satisfy Condition* B, *and let Z: $C^n \to \mathbb{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathcal{T}$ satisfying Condition* C. *Let τ be an s-complete simplex in $A(s)$, which is a face of a simplex in $\mathcal{T}$. If $\tau = \{0^n\}$ or $\tau = \{1^n\}$, then there exists exactly one adjacent complete simplex to τ, being a face of a simplex in $\mathcal{T}$. Otherwise, τ has exactly two adjacent complete simplices, which are faces of simplices in $\mathcal{T}$. Moreover, there exists a finite sequence of simplices $\tau^0, \dots, \tau^M$ such that $\tau^0 = \{0^n\}$, $\tau^M = \{1^n\}$, τ^{k-1} and τ^k are adjacent complete simplices, and τ^k is a face of a simplex in $\mathcal{T}$, $\forall k \in I_M$.*

PROOF. By Lemma 3.2, $\tau = \{0^n\}$ is an s-complete simplex in $A(s)$ if and only if $s = s^0$ where $s^0 = (+1^{n-1^\top}, 0)^\top$. Since $\mathcal{T}$ is a triangulation and $\{0^n\}$ is a facet in the boundary of $A(s^0)$, there is a unique 1-simplex σ in $A(s^0)$ such that $\{0^n\}$ is a facet of σ. By Lemma 3.5 either σ is an $\bar{s}$-complete simplex in $A(\bar{s})$ for some sign vector $\bar{s}$ with $|I^0(\bar{s})| = 2$ and has no other s^0-complete facets, or σ has exactly one other s^0-complete facet and is not an $\bar{s}$-complete simplex in $A(\bar{s})$ for any sign vector $\bar{s}$. Hence, there exists exactly one adjacent complete simplex to $\{0^n\}$. The argument for $\tau = \{1^n\}$ is similar.

Now let τ be an s-complete $(t-1)$-simplex in $A(s)$ for some sign vector s where τ is not equal to $\{0^n\}$ or $\{1^n\}$. There are two possibilities, either τ lies in the relative boundary of $A(s)$ or τ lies in the relative interior of $A(s)$.

If τ lies in the relative boundary of $A(s)$, then, by the properties of a triangulation, there is a unique t-simplex σ in $A(s)$ having τ as a facet. By Lemma 3.5 either σ is s'-complete in $A(s')$ for precisely one sign vector s' and has no other s-complete facets, or σ has exactly one other s-complete facet. This yields one adjacent complete simplex to τ. Since τ lies in the relative boundary of $A(s)$, τ lies in $A(\bar{s})$ for a unique sign vector $\bar{s}$ with $|I^0(\bar{s})| = t - 1$. By Lemma 3.6 either τ is $\bar{s}$-complete in $A(\bar{s})$ for a unique sign vector $\bar{s} \neq s$ with $|I^0(\bar{s})| = t$ and has no $\bar{s}$-complete facets, or τ has exactly one $\bar{s}$-complete facet. In the latter case we are done. In the former case we use the fact that τ is in the relative boundary of $A(\bar{s})$ and hence there is exactly one simplex $\bar{\sigma}$ in $A(\bar{s})$ having τ as a facet. Applying Lemma 3.5 again yields that either $\bar{\sigma}$ is an $\hat{s}$-complete simplex in $A(\hat{s})$ and has no other $\bar{s}$-complete facets, or $\bar{\sigma}$ has exactly one other $\bar{s}$-complete facet. This shows that τ has exactly two adjacent complete simplices.

If τ lies in the relative interior of $A(s)$, then by the properties of a triangulation, it holds that τ is a facet of exactly two simplices in $A(s)$. Applying Lemma 3.5 twice shows that τ has exactly two adjacent complete simplices.

Now let $\tau^0 = \{0^n\}$. Let τ^1 be the unique adjacent complete simplex to τ^0. Given τ^k for some $k \geq 1$ which is not equal to $\{1^n\}$, there exists a unique adjacent complete simplex τ^{k+1} which is not equal to τ^{k-1}. Now the "door-in door-out" argument of Lemke and Howson (1964) excludes cycling and the finiteness of the number of faces of the simplices in the triangulation $\mathcal{T}$ yields that in a finite number of steps, say M, the simplex τ^M must be equal to $\{1^n\}$. This simplex has no other adjacent complete simplex but τ^{M-1}. Q.E.D.

Now we can give a detailed description of the steps of the algorithm generating the simplices $\tau^0, \dots, \tau^M$ given in Theorem 3.7.

ALGORITHM

Step 0. Let $i = 0, t = 1, x^1 = 0^n, \tau^i = \tau(x^1), s = (+1^{n-1^\top}, 0)^\top$, and let x^{t+1} be the unique vertex of the simplex in $A(s)$ containing τ^0 as a facet opposite to it.

Step 1. Let σ be equal to the convex hull of x^{t+1} and τ^i. Pivot $(0, Z(x^{t+1})^\top)^\top$ lexicographically into the linear system (2) corresponding to A_{s,τ^i}, yielding as described in Lemma 3.5, a unique column $k \in I_n$ of A_{s,τ^i} which has to be replaced. If $k \in I_n \setminus I_t$, then go to Step 3 with $j' = i_{k-t}$.

Step 2. Set $i = i + 1$ and let τ' be the facet of σ opposite the vertex x^k. If $\tau' = \{1^n\}$, then the algorithm terminates. If τ' lies in $A(\tilde{s})$ for some $\tilde{s}$ with $t - 1$ zero components, then go to Step 4. Otherwise, there is exactly one simplex $\tilde{\sigma}$ in $A(s)$ such that $\tilde{\sigma} \neq \sigma$ and τ' is a facet of $\tilde{\sigma}$. Go to Step 1 with x^{t+1} as the unique vertex in $\tilde{\sigma}$ opposite to the facet τ'.

Step 3. Define $\tilde{s}$ by $\tilde{s}_{j'} = 0$ and $\tilde{s}_j = s_j$, $\forall j \in I_n \setminus \{j'\}$. There is a unique simplex $\tilde{\sigma}$ in $A(\tilde{s})$ having σ as a facet. Set $i = i + 1, t = t + 1$, and go to Step 1 with x^{t+1} as the unique vertex in $\tilde{\sigma}$ opposite to σ, $s = \tilde{s}$, and $\tau' = \sigma$.

Step 4. Let σ be equal to τ'. Pivot $(0, \tilde{s}_{\tilde{k}} e(\tilde{k})^\top)^\top$ lexicographically into the linear system (2) determined by $A_{s,\tau'}$, where $\tilde{k} \in I_n$ is such that $s_{\tilde{k}} = 0$ and $\tilde{s}_{\tilde{k}} \neq 0$. By Lemma 3.6 there is a unique column $k \in I_n$ of $A_{s,\tau'}$ which has to be replaced. If $k \in I_n \setminus I_t$, then go to Step 3 with $j' = i_{k-t}$, $s = \tilde{s}$, $t = t - 1$, and $i = i - 1$. Otherwise, go to Step 2 with $s = \tilde{s}$ and $t = t - 1$.

It is worthwhile to mention that we can also start the algorithm with the simplex $\{1^n\}$ and terminate with the simplex $\{0^n\}$. In Theorem 3.8 it is shown that the sequence of adjacent simplices generated by the algorithm yields a piecewise linear path of points in C^n connecting 0^n and 1^n such that every point q on the path satisfies (1), i.e., for some $\beta \in \mathbb{R}$ it holds that $0 \leq Z_j(q) \leq \beta$ if $q_j = 0$, $Z_j(q) = \beta$ if $0 < q_j < 1$, and $0 \geq Z_j(q) \geq \beta$ if $q_j = 1$.

THEOREM 3.8. *Let $\mathcal{T}$ be a triangulation of C^n and let the correspondence ζ: $C^n \to \mathbb{R}^n$ satisfy Condition* B. *Let Z: $C^n \to \mathbb{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathcal{T}$ satisfying Condition* C. *Then there exists a continuous, piecewise linear function f: $[0,1] \to C^n$ satisfying $f(0) = 0^n$, $f(1) = 1^n$, and for every $r \in [0,1]$ it holds that for some $\beta \in \mathbb{R}$,*

$$0 \leq Z_j(f(r)) \leq \beta \quad \text{if } f_j(r) = 0,$$

$$Z_j(f(r)) = \beta \quad \text{if } 0 < f_j(r) < 1,$$

$$0 \geq Z_j(f(r)) \geq \beta \quad \text{if } f_j(r) = 1.$$

PROOF. Consider all different pairs of sign vectors and simplices $(s^0, \tau^0), \dots, (s^{M'}, \tau^{M'})$ successively generated by the algorithm, satisfying τ^k is s^k-complete in $A(s^k)$, $\forall k \in \{0\} \cup I_{M'}$, and let $t^k = |I^0(s^k)|$. Notice that $M' \geq M$, where M is as in Theorem 3.7, with $M' > M$ only if Case 2 of Lemma 3.6, which corresponds to the first case in Step 4 of the algorithm, occurs during the algorithm. Clearly, $A^{-1}_{s^k,\tau^k}$ is semi-lexicopositive for every $k \in \{0\} \cup I_{M'}$. Consequently, $\sum_{j=1}^{n+1} (A_{s^k,\tau^k})_{\cdot j} (A^{-1}_{s^k,\tau^k})_{j1} = (1, 0^{n\top})^\top$, or equivalently

$$\sum_{j=1}^{t^k} (A^{-1}_{s^k,\tau^k})_{j1} \begin{pmatrix} 1 \\ Z(x^j) \end{pmatrix} + \sum_{j=1}^{n-t^k} (A^{-1}_{s^k,\tau^k})_{t^k+j,1} \begin{pmatrix} 0 \\ s^k_{i_j} e(i_j) \end{pmatrix}$$

$$+ (A^{-1}_{s^k,\tau^k})_{n+1,1} \begin{pmatrix} 0 \\ -1^n \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix},$$

where $\tau^k = \tau^k(x^1, \dots, x^{t^k})$ and $(A^{-1}_{s^k,\tau^k})_{j1} \geq 0$, $\forall j \in I_n$. Let $y^k = \sum_{j=1}^{t^k} (A^{-1}_{s^k,\tau^k})_{j1} x^j$ and define f: $[0,1] \to C^n$ by

$$f(r) = (1 - M'r + \lfloor M'r \rfloor) y^{\lfloor M'r \rfloor} + (M'r - \lfloor M'r \rfloor) y^{\lfloor M'r \rfloor + 1}, \qquad \forall r \in [0,1],$$

where for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x, and $y^{M'+1}$ is an arbitrary vector in $\mathbb{R}^n$. This function f satisfies the requirements of the theorem. To verify this, remark that for every $k \in I_{M'}$, $A_{s^{k-1},\tau^{k-1}}$ and A_{s^k,τ^k} have n columns in common. Let $B_{\bar{s}^{k-1},\sigma^{k-1}}$ be the $(n+1) \times (n+2)$ matrix which contains all the columns of $A_{s^{k-1},\tau^{k-1}}$ and A_{s^k,τ^k}, $\forall k \in I_{M'}$. The matrix $B_{\bar{s}^{k-1},\sigma^{k-1}}$ yields the system as in (2) for a simplex σ^{k-1} which is the convex hull of τ^{k-1} and τ^k and a sign vector $\bar{s}^{k-1}$ which is such that $I^-(\bar{s}^{k-1}) = I^-(s^{k-1}) \cup I^-(s^k)$ and $I^+(\bar{s}^{k-1}) = I^+(s^{k-1}) \cup I^+(s^k)$. It is easily verified that σ^{k-1} is a simplex in $A(\bar{s}^{k-1})$. The first columns of both $A^{-1}_{s^{k-1},\tau^{k-1}}$ and $A^{-1}_{s^k,\tau^k}$, extended with a zero component, yield admissible solutions to the system in (2) induced by $B_{\bar{s}^{k-1},\sigma^{k-1}}$. Finally, when $\tilde{y}$, $\bar{y} \in \mathbb{R}^{n+2}$ correspond both to admissible solutions to the system $B_{\bar{s}^{k-1},\sigma^{k-1}}y = (1, 0^{n^\top})^\top$, then $\lambda\tilde{y} + (1-\lambda)\bar{y}$ also corresponds to an admissible solution for every $\lambda \in [0,1]$. Since for every $k \in I_{M'}$ the simplex σ^{k-1} lies in $A(\bar{s}^{k-1})$, it follows from (2) and Condition C that for every $r \in [0,1]$ the point $q = f(r)$ satisfies (1). Q.E.D.

4. Accuracy analysis.

First it will be argued in Theorem 4.1 and Corollary 4.2 that the points lying on the path given in Theorem 3.8 indeed all correspond to approximate zero points of ζ. To show this, a sequence of triangulations $\mathcal{T}^r$ with mesh size converging to zero is taken. This yields according to Theorem 3.8, for every $r \in \mathbb{N}$, a continuous piecewise linear function f^r: $[0,1] \to C^n$ with image set $f^r([0,1])$ connecting 0^n and 1^n. It will be shown that if q^r is an arbitrary point in $f^r([0,1])$ and the sequence $(q^r)_{r\in\mathbb{N}}$ converges to q, then $0^n \in \zeta(q)$. It should be remarked that by the compactness of C^n, every sequence of points in C^n has a converging subsequence. Hence, for every $\epsilon > 0$, there exists a number $R \in \mathbb{N}$ such that for all $r \geq R$ it holds that $q^r \in f^r([0,1])$ implies $\|Z^r(q^r)\|_\infty < \epsilon$, or equivalently $\max_{q^r \in f^r([0,1])}\|Z^r(q^r)\|_\infty \to 0$, where Z^r: $C^n \to \mathbb{R}^n$ is the piecewise linear approximation of ζ with respect to $\mathcal{T}^r$. Next, it will be shown in Theorem 4.3 by a limiting argument that there exists a connected set of zero points of ζ, containing 0^n and 1^n, that is being approximated. In the remainder of the section some results concerning the accuracy of the approximation are discussed.

THEOREM 4.1. *Let ζ: $C^n \to \mathbb{R}^n$ be a correspondence satisfying Condition* B. *For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of C^n with mesh size smaller than $1/r$ and let Z^r: $C^n \to \mathbb{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathcal{T}^r$ satisfying Condition* C. *Let $(q^r)_{r\in\mathbb{N}}$ be an arbitrary convergent sequence of points in C^n with limit q^* such that for any $r \in \mathbb{N}$ it holds that $q^r \in f^r([0,1])$, where f^r corresponds to the function f of Theorem* 3.8 *for the piecewise linear approximation Z^r. Then $0^n \in \zeta(q^*)$.*

PROOF. Let $(\lambda_1^r, \ldots, \lambda_{n+1}^r, x^{1^r}, \ldots, x^{n+1^r}, z^{1^r}, \ldots, z^{n+1^r})_{r\in\mathbb{N}}$ be a sequence of points in $\prod_{i=1}^{n+1}[0,1] \times \prod_{i=1}^{n+1} C^n \times \prod_{i=1}^{n+1} \mathbb{R}^n$ satisfying $\sum_{j=1}^{n+1}\lambda_j^r = 1$, $\sigma(x^{1^r}, \ldots, x^{n+1^r})$ is a simplex of $\mathcal{T}^r$, $q^r = \sum_{j=1}^{n+1}\lambda_j^r x^{j^r} \in f^r([0,1])$, and $z^{j^r} = Z^r(x^{j^r})$. Notice that it may happen that $\lambda_j^r = 0$ for some $j \in I_{n+1}$. By definition, $Z^r(q^r) = \sum_{j=1}^{n+1}\lambda_j^r z^{j^r}$. Define $z^r = Z^r(q^r)$; then $z^r = \beta^r 1^n - \sum_{j\in\{k\in I_n|q_k^r=0\}}\mu_j^r e(j) + \sum_{j\in\{k\in I_n|q_k^r=1\}}\mu_j^r e(j)$, for some $\beta^r \in \mathbb{R}$ and some $\mu_j^r \geq 0$, $\forall j \in \{k \in I_n | q_k^r = 0$ or $q_k^r = 1\}$. Since $\bigcup_{q\in C^n}\zeta(q)$ is bounded, the sequence given above remains in a compact set, and without loss of generality it can be assumed to converge to an element $(\lambda_1^*, \ldots, \lambda_{n+1}^*, x^{*1}, \ldots, x^{*n+1}, z^{*1}, \ldots, z^{*n+1})$. Define $z^* = \sum_{j=1}^{n+1}\lambda_j^* z^{*j}$. Clearly, it holds that $z^r \to z^*$. Since for every $r \in \mathbb{N}$ the mesh size of $\mathcal{T}^r$ is smaller than $1/r$, it holds for every $j \in I_{n+1}$ that $x^{*j} = q^*$. Using that ζ is upper semi-continuous and $\bigcup_{q\in C^n}\zeta(q)$ is bounded this implies that for every $j \in I_{n+1}$, $z^{*j} \in \zeta(q^*)$. Since ζ is convex valued, $\sum_{j=1}^{n+1}\lambda_j^* = 1$, and $\lambda_j^* \geq 0$, $\forall j \in I_{n+1}$, it holds that $z^* \in \zeta(q^*)$.

If there is a subsequence $(q^{r^s})_{s \in \mathbb{N}}$ of $(q^r)_{r \in \mathbb{N}}$ such that for every $s \in \mathbb{N}$, $0 < q_j^{r^s} < 1$, $\forall j \in I_n$, then $z^{r^s} = \beta^{r^s} 1^n$. Since $z^* \in \zeta(q^*)$, there is some $p \in \mathbb{R}^n_{++}$ such that $p \cdot z^* = 0$. Consequently, $z^* = 0^n$. If there is not such a subsequence, then there exists a subsequence $(q^{r^s})_{s \in \mathbb{N}}$ of $(q^r)_{r \in \mathbb{N}}$ such that for some $k \in I_n$, for every $s \in \mathbb{N}$, $q_k^{r^s} = 0$, or for some $k \in I_n$, for every $s \in \mathbb{N}$, $q_k^{r^s} = 1$. In the first case, using that Z^r satisfies Condition C, it holds that $0 \leq z_k^{r^s} \leq \beta^{r^s}$ and therefore $0^n \leq z^{r^s}$, $\forall s \in \mathbb{N}$. This implies that $z^* \geq 0^n$. In the second case, using again that Z^r satisfies Condition C, it holds that $0 \geq z_k^{r^s} \geq \beta^{r^s}$ and therefore $0^n \geq z^*$. In both cases the existence of a $p \in \mathbb{R}^n_{++}$ such that $p \cdot z^* = 0$ implies that $z^* = 0^n$. Hence, $0^n \in \zeta(q^*)$. Q.E.D.

From Theorem 4.1 the next result immediately follows.

COROLLARY 4.2. *Let ζ: $C^n \to \mathbb{R}^n$ be a correspondence satisfying Condition* B. *For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of C^n with mesh size smaller than $1/r$ and let Z^r: $C^n \to \mathbb{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathcal{T}^r$ satisfying Condition* C. *Then for every $\epsilon > 0$ there exists an $R \in \mathbb{N}$ such that for every $r \geq R$ it holds that $q^r \in f^r([0,1])$ implies $\|Z^r(q^r)\|_\infty < \epsilon$.*

PROOF. Suppose a sequence $(q^r, Z^r(q^r))_{r \in \mathbb{N}}$ exists with $q^r \in f^r([0,1])$ and $\|Z^r(q^r)\|_\infty \geq \epsilon$ for every $r \in \mathbb{N}$. Since C^n is compact and $\bigcup_{q \in C^n} \zeta(q)$ is bounded, there exists a converging subsequence $(q^{r^s}, Z^{r^s}(q^{r^s}))_{s \in \mathbb{N}}$, with limit say (q^*, z^*), where $\|z^*\|_\infty \geq \epsilon > 0$. As in the proof of Theorem 4.1 it can be shown that $z^* = 0^n$, yielding a contradiction. Q.E.D.

Next we show the existence of specific types of constrained equilibria. To do so, consider a sequence of triangulations $(\mathcal{T}^r)_{r \in \mathbb{N}}$ of C^n with mesh size converging to zero. For all $r \in \mathbb{N}$, let Z^r be a piecewise linear approximation of ζ with respect to $\mathcal{T}^r$ satisfying Condition C and let the function f^r be given as in Theorem 3.8. Moreover, let a commodity $j \in I_n$ and some constant $\alpha \in [0,1]$ be given. Since f^r is a continuous function, $f^r(0) = 0^n$, and $f^r(1) = 1^n$, there exists $t^r \in [0,1]$ such that $f_j^r(t^r) = \alpha$. Now define the sequence $(q^r)_{r \in \mathbb{N}}$ by $q^r = f^r(t^r)$. Without loss of generality it can be assumed that $q^r \to q^*$, for some $q^* \in C^n$. By Theorem 4.1 it holds that $0^n \in \zeta(q^*)$. Clearly, $q_j^* = \alpha$. Hence, for every $\alpha \in [0,1]$, there exists a vector $q^\alpha \in C^n$ such that $q_j^\alpha = \alpha$ and $0^n \in \zeta(q^\alpha)$. Now consider the case where the correspondence ζ is the excess demand correspondence of an economy with price rigidities $E = (\{X^i, \succsim^i, w^i\}_{i=1}^m, P)$ and rationing system $(\tilde{l}, \tilde{L})$. If $\underline{p}_j = \bar{p}_j = 1$, we obtain the model of Drèze (1975) where commodity j denotes the numeraire commodity. His equilibrium concept, the so-called Drèze equilibrium, corresponds to an element $q \in C^n$ such that $0^n \in \zeta(q)$ and $q_j = \frac{1}{2}$. The Drèze equilibrium is therefore a constrained equilibrium without rationing on the market of the numeraire commodity. By taking $\alpha = \frac{1}{2}$, it follows that on the path generated by the algorithm there is an approximate Drèze equilibrium. Another equilibrium concept is given by van der Laan (1980). His equilibrium concept, the so-called supply constrained equilibrium, corresponds to an element $q \in C^n$ such that $0^n \in \zeta(q)$ and $\max_{j \in I_n} q_j = \frac{1}{2}$. The supply constrained equilibrium is therefore a constrained equilibrium without rationing on the market of at least one commodity and without demand rationing on every market. We claim that there is an approximate supply constrained equilibrium on the path generated by the algorithm. Consider the set $S = \{q \in C^n | \max_{j \in I_n} q_j = \frac{1}{2}\}$. The element $q \in C^n$ corresponds to a supply constrained equilibrium if $q \in S$ and $0^n \in \zeta(q)$. Since f^r is a continuous function, $f^r(0) = 0^n$ and $f^r(1) = 1^n$, there exists $t^r \in [0,1]$ such that $f^r(t^r) \in S$. Now define the sequence $(q^r)_{r \in \mathbb{N}}$ by $q^r = f^r(t^r)$. Without loss of generality it can be assumed that $q^r \to q^*$. By Theorem 4.1 it holds that $0^n \in \zeta(q^*)$. Obviously, $q^* \in S$.

Using Theorem 3.8 and Theorem 4.1 it will be shown, using the approach taken in Herings (1993), that there exists a connected set C such that $0^n \in C$, $1^n \in C$, and

$0^n \in \zeta(q)$, $\forall q \in C$. Hence, there is a continuum of zero points of ζ being approximated by the algorithm of §3. For a nonempty, compact set $S \subset \mathbb{R}^n$, define the continuous function d_S: $\mathbb{R}^n \to \mathbb{R}$ by $d_S(x) = \min\{\|x - y\|_\infty | y \in S\}$. This function will be used in the proof of Theorem 4.3.

THEOREM 4.3. *Let* ζ: $C^n \to \mathbb{R}^n$ *be a correspondence satisfying Condition* B. *Then there exists a connected set of points* C *in* C^n *such that* $0^n \in C$, $1^n \in C$, *and* $0^n \in \zeta(q)$, $\forall q \in C$.

PROOF. Define $Q = \{q \in C^n | 0^n \in \zeta(q)\}$. Clearly, $0^n \in Q$, $1^n \in Q$, and Q is compact. Suppose the theorem is false. Then 1^n is not an element of the component of Q containing 0^n. By Munkres (1975, p. 235) it holds for every compact set X in some Euclidean space and for every element $x \in X$ that the component of X containing x equals the intersection of all sets containing x which are both open and closed in X. Hence, there exists a set Q^0, which is open and closed in Q, such that $0^n \in Q^0$ and $1^n \notin Q^0$. Define $Q^1 = Q \setminus Q^0$. Then Q^1 is open and closed in Q, $0^n \notin Q^1$, and $1^n \in Q^1$. Since Q is compact, it follows that Q^0 and Q^1 are disjoint, compact sets. Hence, there exists $\epsilon > 0$ such that $\min\{\|q^0 - q^1\|_\infty | q^0 \in Q^0, q^1 \in Q^1\} \geq \epsilon$. For every $r \in \mathbb{N}$, let $\mathscr{T}^r$ be a triangulation of C^n with mesh size smaller than $1/r$, let Z^r: $C^n \to \mathbb{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathscr{T}^r$ satisfying Condition C, and let f^r correspond to the function f of Theorem 3.8 for the piecewise linear approximation Z^r. Define g^r: $[0,1] \to \mathbb{R}$ by

$$g^r(t) = d_{Q^0}(f^r(t)) - d_{Q^1}(f^r(t)), \qquad \forall t \in [0,1].$$

Since g^r is continuous, $g^r(0) \leq -\epsilon$, and $g^r(1) \geq \epsilon$, there exists a point $t^r \in [0,1]$ such that $g^r(t^r) = 0$. Hence, $d_{Q^0}(f^r(t^r)) = d_{Q^1}(f^r(t^r)) = d_Q(f^r(t^r)) \geq \frac{1}{2}\epsilon$. Without loss of generality it can be assumed that $(f^r(t^r))_{r \in \mathbb{N}}$ converges to a point $q^* \in C^n$. Hence,

$$d_Q(q^*) = d_Q\Big(\lim_{r\to\infty} f^r(t^r)\Big) = \lim_{r\to\infty} d_Q(f^r(t^r)) \geq \frac{1}{2}\epsilon > 0.$$

However, by Theorem 4.1, $d_Q(q^*) = 0$, yielding a contradiction. Q.E.D.

For the remainder of this section, consider the case where ζ satisfying Condition B is a continuous function, denoted by z. Then there exists a function p: $C^n \to \mathbb{R}^n_{++}$ (not necessarily continuous) satisfying $p(q) \cdot z(q) = 0$. We will now derive some properties of the solutions generated by the algorithm. Let $\epsilon > 0$ be given and let $\delta > 0$ be such that for $\tilde{q}, \bar{q} \in C^n, \|\tilde{q} - \bar{q}\|_\infty < \delta$ implies $\|z(\tilde{q}) - z(\bar{q})\|_\infty < \epsilon$. By the continuity of z and the compactness of C^n such a δ exists. Consider a triangulation $\mathscr{T}$ with mesh size less than δ and suppose q lies in the image set $f([0,1])$ of the piecewise linear function f given by Theorem 3.8. Then there exist numbers $\lambda_j \geq 0$, $\forall j \in I_{n+1}$, $\mu_j \geq 0$, $\forall j \in I_n$, and $\beta \in \mathbb{R}$ such that $\sum_{j=1}^{n+1} \lambda_j = 1, q = \sum_{j=1}^{n+1} \lambda_j x^j$ for vertices x^j of a simplex in $\mathscr{T}$ containing q, and, for every $j \in I_n$,

$$Z_j(q) = \beta - \mu_j \quad \text{if } q_j = 0,$$

$$Z_j(q) = \beta \quad \text{if } 0 < q_j < 1,$$

$$Z_j(q) = \beta + \mu_j \quad \text{if } q_j = 1.$$

Clearly, $\|Z(q) - z(q)\|_\infty = \|\sum_{j=1}^{n+1} \lambda_j (z(x^j) - z(q))\|_\infty < \epsilon$. Hence, $Z(q) - \epsilon 1^n \ll z(q) \ll Z(q) + \epsilon 1^n$. If for some $j \in I_n$, $q_j = 0$, then $\beta - \mu_j = Z_j(q) \geq 0$, so $\beta \geq \mu_j \geq 0$. If for some $j \in I_n$, $q_j = 1$, then $\beta + \mu_j = Z_j(q) \leq 0$, so $\beta \leq -\mu_j \leq 0$.

First consider the case where for some $j \in I_n$, $q_j = 0$, and for some $j \in I_n$, $q_j = 1$. Then $\beta = 0$ and $\mu_j = 0$, $\forall j \in I_n$ satisfying $q_j = 0$ or $q_j = 1$. So, $Z(q) = 0^n$ and therefore $-\epsilon 1^n \ll z(q) \ll \epsilon 1^n$. Moreover, if $q_j = 0$ then $0 \leq z_j(q) < \epsilon$, and if $q_j = 1$ then $-\epsilon < z_j(q) \leq 0$.

Consider now the case where for every $j \in I_n$, $0 < q_j < 1$. So, $Z(q) = \beta 1^n$. Then

$$|\beta| \sum_{j=1}^{n} p_j(q) = |p(q) \cdot Z(q)| = |p(q) \cdot (Z(q) - z(q))| < \epsilon \sum_{j=1}^{n} p_j(q).$$

So, $|\beta| < \epsilon$. Consequently, $(\beta - \epsilon)1^n \ll z(q) \ll (\beta + \epsilon)1^n$ with $|\beta| < \epsilon$.

Next consider the case where for some $j \in I_n$, $q_j = 1$, and for every $j \in I_n$, $q_j > 0$. If $q_j = 1$ then $\beta + \mu_j \leq 0$, so it holds that $\beta \leq 0$ and $0 \leq \mu_j \leq -\beta$. Moreover,

$$0 = p(q) \cdot z(q) < (\beta + \epsilon) \sum_{j \in \{k \in I_n | 0 < q_k < 1\}} p_j(q).$$

Therefore, $\beta + \epsilon > 0$ and $-\epsilon < \beta \leq 0$. Hence,

$$\begin{aligned} \beta - \epsilon < z_j(q) &\leq 0 && \text{if } q_j = 1, \\ \beta - \epsilon < z_j(q) &< \beta + \epsilon && \text{if } 0 < q_j < 1, \end{aligned}$$

with $-\epsilon < \beta \leq 0$.

Finally, consider the case where for some $j \in I_n$, $q_j = 0$, and for every $j \in I_n$, $q_j < 1$. As in the previous paragraph, it can be shown that

$$\begin{aligned} 0 \leq z_j(q) &< \beta + \epsilon && \text{if } q_j = 0, \\ \beta - \epsilon < z_j(q) &< \beta + \epsilon && \text{if } 0 < q_j < 1, \end{aligned}$$

with $0 \leq \beta < \epsilon$.

All cases are summarized in Theorem 4.4.

THEOREM 4.4. *Let* $z\colon C^n \to \mathbb{R}^n$ *be a function satisfying Condition* B. *Let* $\epsilon > 0$ *be given and choose* $\delta > 0$ *such that for all* $\tilde{q}, \bar{q} \in C^n$, $\|\tilde{q} - \bar{q}\|_\infty < \delta$ *implies* $\|z(\tilde{q}) - z(\bar{q})\|_\infty < \epsilon$. *Let* $\mathcal{T}$ *be a triangulation with mesh size less than* δ *and let* $q \in f([0, 1])$ *with* f *as in Theorem* 3.8. *Then there is a* $\beta \in \mathbb{R}$ *satisfying* $-\epsilon < \beta < \epsilon$, *with* $0 \leq \beta$ *if* $q_j = 0$ *for some* $j \in I_n$, *and* $\beta \leq 0$ *if* $q_j = 1$ *for some* $j \in I_n$, *such that*

$$\begin{aligned} 0 \leq z_j(q) &< \beta + \epsilon && \text{if } q_j = 0, \\ \beta - \epsilon < z_j(q) &< \beta + \epsilon && \text{if } 0 < q_j < 1, \\ \beta - \epsilon < z_j(q) &\leq 0 && \text{if } q_j = 1. \end{aligned}$$

5. An illustration of the algorithm. In this section we will illustrate the algorithm by an example of a correspondence ζ satisfying Condition B. We choose the example such that it is possible to determine analytically all elements $q \in C^n$ satisfying $0^n \in \zeta(q)$, and therefore it is possible to compare the set of approximate zero points generated by the algorithm with the exact zero points of ζ. The correspondence ζ in the example is the excess demand function z of an economy with price rigidities, $E = (\{X^i, \succcurlyeq^i, w^i\}_{i=1}^2, P)$, where $X^1 = X^2 = \mathbb{R}^2_+$. This economy consists of 2 consumers and 2 commodities. The preferences $\succcurlyeq^1$ and $\succcurlyeq^2$ can be represented by utility functions given by $u^1(x_1, x_2) = (x_1)^{3/4}(x_2)^{1/4}$, $\forall x \in \mathbb{R}^2_+$, and $u^2(x_1, x_2) = (x_1)^{1/4}(x_2)^{3/4}$, $\forall x \in \mathbb{R}^2_+$, respectively, $w^1 = (1,4)^\top$, $w^2 = (2,1)^\top$, and $P = \{p \in \mathbb{R}^2_+ | \frac{1}{6} \le p_1 \le 2, p_2 = 1\}$. The functions $\tilde{l}$: $C^2 \to -\mathbb{R}^4_+$ and $\tilde{L}$: $C^2 \to \mathbb{R}^4_+$ are given by

$$\begin{aligned} \tilde{l}^1_1(q) = \tilde{l}^2_1(q) &= -3q_1, \ \forall q \in C^2, \\ \tilde{l}^1_2(q) = \tilde{l}^2_2(q) &= -5q_2, \ \forall q \in C^2, \\ \tilde{L}^1_1(q) = \tilde{L}^2_1(q) &= 18q_1, \quad \forall q \in C^2, \\ \tilde{L}^1_2(q) = \tilde{L}^2_2(q) &= 5q_2, \quad \forall q \in C^2. \end{aligned}$$

Now the price and rationing function $(\hat{p}, \hat{l}, \hat{L})$: $C^2 \to P \times -\mathbb{R}^4_+ \times \mathbb{R}^4_+$ is easily obtained. It can be derived that the excess demand function of consumer 1 is given by

$$z^1(q) = \begin{cases} (90q_2, -15q_1)^\top, & 0 \le q_1 \le \frac{1}{3}, 0 \le q_2 \le \frac{71}{360}, \\ \left(\frac{71}{4}, -\frac{71}{24}\right)^\top, & 0 \le q_1 \le \frac{1}{3}, \frac{71}{360} \le q_2 \le 1, \\ \left(\frac{90q_2}{33q_1 - 10}, -15q_2\right)^\top, & \frac{1}{3} \le q_1 \le \frac{2}{3}, 0 \le q_2, 33q_1 + 360q_2 \le 82, \\ \left(\frac{82 - 33q_1}{132q_1 - 40}, \frac{33q_1 - 82}{24}\right)^\top, & \frac{1}{3} \le q_1 \le \frac{2}{3}, q_2 \le 1, 33q_1 + 360q_2 \ge 82, \\ \left(\frac{15q_2}{2}, -15q_2\right)^\top, & \frac{2}{3} \le q_1, 0 \le q_2 \le \frac{1}{6}, 36q_1 + 5q_2 \le 36, \\ \left(\frac{5}{4}, -\frac{5}{2}\right)^\top, & \frac{2}{3} \le q_1 \le \frac{211}{216}, \frac{1}{6} \le q_2 \le 1, \\ (54 - 54q_1, 108q_1 - 108)^\top, & \frac{211}{216} \le q_1 \le 1, q_2 \le 1, 36q_1 + 5q_2 \ge 36. \end{cases}$$

The excess demand function of consumer 2 is given by

$$z^2(q) = \begin{cases} (0,0)^\top, & 0 \le q_1 \le \frac{1}{3}, 0 \le q_2 \le 1, \\ \left(\frac{33 - 99q_1}{66q_1 - 20}, \frac{33q_1 - 11}{4}\right)^\top, & \frac{1}{3} \le q_1 \le \frac{2}{3}, 0 \le q_2, 33q_1 + 60q_2 \le 71, \\ \left(\frac{90q_2 - 90}{33q_1 - 10}, 15 - 15q_2\right)^\top, & q_1 \le \frac{2}{3}, q_2 \le 1, 33q_1 + 60q_2 \ge 71, \\ \left(-\frac{11}{8}, \frac{11}{4}\right)^\top, & \frac{2}{3} \le q_1 \le 1, 0 \le q_2 \le \frac{49}{60}, \\ \left(\frac{15q_2 - 15}{2}, 15 - 15q_2\right)^\top, & \frac{2}{3} \le q_1 \le 1, \frac{49}{60} \le q_2 \le 1. \end{cases}$$

Now the excess demand function, z: $C^2 \to \mathbb{R}^2$, is given by the function $z = z^1 + z^2$. The zero points of z can easily be determined analytically. It can be verified that the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{148}{231} \\ \frac{71}{420} \end{pmatrix}, \begin{pmatrix} \frac{148}{231} \\ \frac{349}{420} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} \frac{211}{216} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

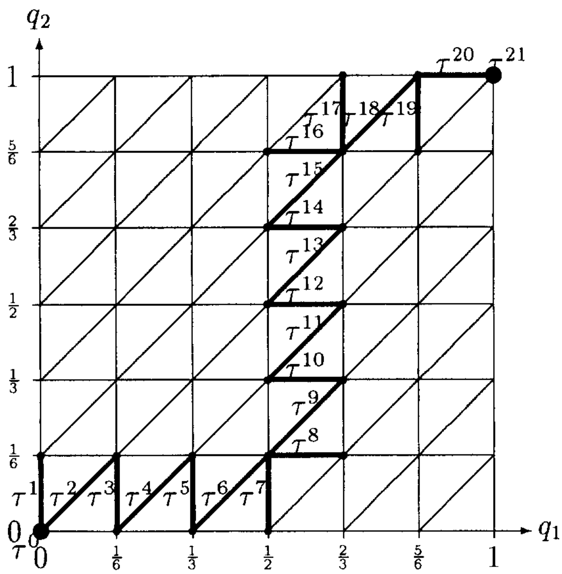

FIGURE 3.

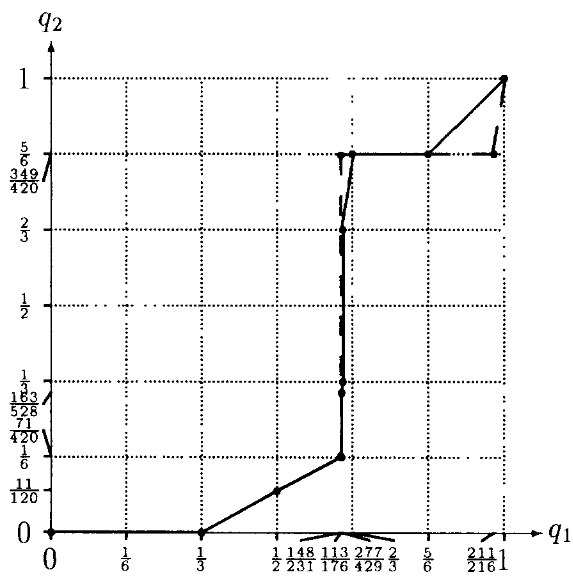

FIGURE 4.

and the convex combinations of any two successive points are all zero points of z. Next, the algorithm is used to compute approximate zero points of z by generating a sequence of adjacent complete simplices for the K-triangulation with mesh size equal to $\frac{1}{6}$. In Figure 3 all adjacent complete 1-simplices are drawn by thick lines. The sequence of adjacent complete simplices $\tau^0, \ldots, \tau^{21}$ corresponds to the one in Theorem 3.8. The piecewise linear path $f([0, 1])$ of points generated by the algorithm is given by the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{11}{120} \end{pmatrix}, \begin{pmatrix} \frac{113}{176} \\ \frac{1}{6} \end{pmatrix}, \begin{pmatrix} \frac{113}{176} \\ \frac{163}{528} \end{pmatrix}, \begin{pmatrix} \frac{277}{429} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{277}{429} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} \frac{5}{6} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and all convex combinations of two successive points. In Figure 4 the solid line corresponds to the piecewise linear path of approximate zero points generated by the algorithm, while the broken line corresponds to the set of zero points of z. The algorithm starts with the s^0-complete simplex $\tau^0 = \{(0,0)^\top\}$ in $A(s^0)$, where $s^0 = (+1,0)^\top$. Notice that although τ^0 is also $(0,-1)^\top$-complete, τ^0 does not lie in $A((0,-1)^\top)$. The simplex τ^0 is not s-complete for $s = (0,+1)^\top$ since $A^{-1}_{(0,+1)^\top,\tau^0}$ is not semi-lexicopositive, although the system $A_{(0,+1)^\top,\tau^0}x = 0$ does have a solution satisfying $x_1 \geq 0$, $x_2 \geq 0$. In this way the lexicographic pivot rules as described in this paper determine in a unique way the direction the algorithm will follow. So, the next vertex which is brought into the system is $(0,\frac{1}{6})^\top$. Remark that the direction determined by the lexicographic pivot rules is orthogonal to the set of zero points of z around $(0,0)^\top$. It can easily be verified that the 0-simplex $\tau((0,\frac{1}{6})^\top)$ is not $(+1,0)^\top$-complete. However, the 1-simplex $\tau^1 = \tau((0,0)^\top,(0,\frac{1}{6})^\top)$ is $(0,0)^\top$-complete and therefore we are in Case 1 of Lemma 3.5. The simplex $\tau^2 = \tau((0,0)^\top, (\frac{1}{6},\frac{1}{6})^\top)$ is the unique $(0,0)^\top$-complete facet of the simplex $\sigma((0,0)^\top,(0,\frac{1}{6})^\top,(\frac{1}{6},\frac{1}{6})^\top)$ in $A((0,0)^\top) = C^2$ which is not equal to τ^1. This corresponds to Case 2 of Lemma 3.5, and so on.

From Figure 4 it can be seen that although the lexicographic pivot rules determine an initial direction to leave τ^0 being orthogonal to the set of zero points of z around $(0,0)^\top$, the points on the piecewise linear path of approximate zero points generated by the simplices $\tau^0,\ldots,\tau^5$ are zero points of z. An interesting situation occurs at the simplex $\tau^{14} = \tau((\frac{1}{2},\frac{2}{3})^\top,(\frac{2}{3},\frac{2}{3})^\top)$. For this simplex it holds that

$$A^{-1}_{(0,0)^\top,\tau^{14}} = \begin{pmatrix} \frac{18}{143} & \frac{48}{143} & -\frac{48}{143} \\ \frac{125}{143} & -\frac{48}{143} & \frac{48}{143} \\ \frac{5}{104} & -\frac{7}{13} & -\frac{6}{13} \end{pmatrix}.$$

The vertex brought into the system is given by $x^3 = (\frac{2}{3},\frac{5}{6})^\top$ with $Z(x^3) = (0,0)^\top$. So, $y = A^{-1}_{(0,0)^\top,\tau^{14}}(1, Z(x^3)^\top)^\top = (\frac{18}{143},\frac{125}{143},\frac{5}{104})^\top$. Now there occurs a degeneracy problem since both y_1 and y_2 are positive and equal to the corresponding element of the first column of $A^{-1}_{(0,0)^\top,\tau^{14}}$. However, since the vector

$$\frac{\left(A^{-1}_{(0,0)^\top,\tau^{14}}\right)_{1\cdot}}{y_1}$$

is lexicographically larger than the vector

$$\frac{\left(A^{-1}_{(0,0)^\top,\tau^{14}}\right)_{2\cdot}}{y_2},$$

the vertex of τ^{14} corresponding to column 2 of $A_{(0,0)^\top,\tau^{14}}$, $(\frac{2}{3},\frac{2}{3})^\top$, should be replaced by the vertex $(\frac{2}{3},\frac{5}{6})^\top$, yielding the simplex τ^{15}. Moreover, it has to be remarked that using the usual lexicographic pivoting rules would imply that the last column of $A_{(0,0)^\top,\tau^{14}}$ should be replaced by the vertex $(\frac{2}{3},\frac{5}{6})^\top$. However, the last column of any $A_{s,\tau}$, corresponding to the unrestricted variable β, never leaves the system. Another interesting case occurs at τ^{20}. The 1-simplex τ^{20} lies in $A((0,-1)^\top)$ and therefore $(0,0,-1)^\top$ is the new column brought into the system. We are in Case 3 of Lemma 3.6, where τ^{21} is the unique $(0,-1)^\top$-complete facet of τ^{20}. The 0-simplex τ^{21} lies in $A((-1,-1)^\top) = \{(1,1)^\top\}$ and we are in Case 1 of Lemma 3.6. The algorithm now terminates with the $(0,-1)^\top$-complete simplex $\{(1,1)^\top\}$. It is clear from Figure 4 that the approximate zero points lying in the s-complete $(t-1)$-simplices in $A(s)$ for some sign vector s with $|I^0(s)| = t$ are everywhere very close to the zero points of z themselves. It should be remarked that this is caused by the fact that the function z given in the example has such an easy structure. For more complicated correspondences a finer mesh size might be needed to obtain a similar accuracy.

The approximate supply constrained equilibrium, being the constrained equilibrium without binding demand rationing and with at least one market without rationing, corresponds to $q^S = (\frac{1}{2}, \frac{11}{120})^\top$, and the excess demand equals zero in this point, $z(q^S) = (0,0)^\top$. This constrained equilibrium is also the Drèze equilibrium if commodity 1 is assumed to be the numeraire commodity as in Drèze's original model. If commodity 2 is assumed to be the numeraire commodity, then the approximate Drèze equilibrium, being also the constrained equilibrium without binding supply rationing and with at least one market without rationing, corresponds to $q^D = (\frac{277}{429}, \frac{1}{2})$. The excess demand in this point is close to zero, $z(q^D) = (-0.0255, 0.0480)^\top$.

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