



# On the asymptotic uniqueness of bargaining equilibria<sup>☆</sup>

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## ABSTRACT

This note reexamines the connection between the asymmetric Nash bargaining solution and the equilibria of strategic bargaining games. A first example shows non-convergence to the asymmetric Nash bargaining solution. A second example demonstrates the possibility of multiple limits.

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## 1. Introduction

The theory of non-cooperative bargaining has resulted in a substantial body of research following the seminal contribution of Rubinstein (1982). We are interested in the generalization of this model to the case with  $n$  players. The main characteristics of the model are that  $n$  players bargain about an  $n$ -dimensional convex set of feasible payoffs, decision making requires the consent of all the players involved, and bargaining takes place in an alternating offers style. At the beginning of each period, some player is selected as the proposer and makes a proposal, followed by an accept or reject decision of the other players. In case of rejection by at least one player, there is some probability that negotiations break down. Otherwise, bargaining proceeds in the next period with the selection of a new proposer, and so on, and so forth.

The existence of a subgame perfect equilibrium in stationary strategies in this model follows from the results of Banks and Duggan (2000). A number of papers show, under increasingly weaker assumptions, that in the limit, when the breakdown probability converges to zero, the equilibrium converges to the Nash bargaining solution. Binmore et al. (1986) show that the unique subgame perfect equilibrium of the 2-player game of bargaining with alternating offers as by Rubinstein (1982) converges to the Nash bargaining solution as the time interval between two consecutive rounds of bargaining

vanishes. The stationary subgame perfect equilibria in the bargaining procedure of Hart and Mas-Colell (1996) converge to the Nash bargaining solution in the  $n$ -player case when applied to the special case where decision making requires unanimous approval, and Kultti and Vartiainen (2010) show convergence to the Nash bargaining solution for the  $n$ -player case when players rotate in making offers.

Miyakawa (2008), Laruelle and Valenciano (2008), Britz et al. (2010) give a non-cooperative foundation for the asymmetric Nash bargaining solution in the case of  $n$  players. Miyakawa (2008) and Laruelle and Valenciano (2008) consider a bargaining game with time-invariant recognition probabilities and show that, as the probability of breakdown of negotiations becomes small, the stationary subgame perfect equilibria approach the asymmetric Nash bargaining solution, with the weights corresponding to the players' recognition probabilities. Britz et al. (2010) establish a similar convergence result in a game where the selection of the proposer is determined by a Markov process, thereby covering both time-invariant recognition probabilities and rotating offers as special cases. The weights in the asymmetric Nash bargaining solution are now given by the probabilities of the stationary distribution of the proposer selection process.

Such convergence results are desirable for at least two reasons. They imply that bargaining equilibria are asymptotically unique and that it is rather straightforward to compute what the limit equilibrium looks like. Apart from asymptotic uniqueness of bargaining equilibrium, there are few general uniqueness results for the  $n$ -player case with unanimous approval. The main result is by Merlo and Wilson (1995), who show that bargaining equilibria are unique when the best response function has a contraction property. The contraction

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property is shown to hold when the set of feasible payoffs is derived from linear utility functions, or from separable concave utility functions with deterministically chosen proposers.

All the papers on multilateral bargaining that give convergence results to the Nash bargaining solution do so under the assumption that the convex set of feasible payoffs has a boundary that is non-level and differentiable. The non-levelness of the boundary means that each weakly Pareto-efficient payoff vector is also Pareto efficient. The assumption of differentiability of the boundary means that the normal vector to the set of feasible payoffs is unique up to a multiplier. Since general convex sets of feasible payoffs can be approximated by differentiable ones, one may conjecture that the asymptotic uniqueness of bargaining equilibria and their convergence to the Nash bargaining solution carries over to the general convex case. This intuition is reinforced by the fact that a general result holds for the case with two players.

Nevertheless, this note demonstrates that the differentiability assumption is crucial. To do so we construct two examples with three players where the payoff set satisfies all of the standard assumptions except for the differentiability assumption. The main feature of our first example is that the unique stationary subgame perfect equilibrium does not converge to the Nash bargaining solution. A similar example, though not satisfying non-levelness, was developed independently by Kultti and Vartiainen (2010), who follow up on a seminal idea due to Lensberg and Thomson (1988). The main feature of the second example is that there are two distinct stationary subgame perfect equilibria that converge to different limits as the breakdown probability becomes small. This example therefore shows that the uniqueness result of Merlo and Wilson (1995) does not hold for general convex sets of feasible payoffs and that asymptotic uniqueness of bargaining equilibria may fail when the set of feasible payoffs has a non-differentiable boundary.

These findings are particularly surprising in light of the existing literature on one-dimensional bargaining. In a one-dimensional bargaining model the players negotiate over the choice of an alternative represented by a point in an interval. In such a game the set of attainable payoffs is one-dimensional and therefore is neither comprehensive nor convex. The asymptotic uniqueness of stationary equilibria has been established for a number of variations of the one-dimensional model of bargaining, see Cho and Duggan (2003), Cardona and Ponsati (2007), Predtetchinski (2007), and Herings and Predtetchinski (2010). In the one-dimensional model, for asymptotic uniqueness, it is important that utility functions be continuous and concave, but may well have points of non-differentiability. Non-differentiability of utility functions gives rise to a kink in the set of feasible payoffs, but this does not destroy the asymptotic uniqueness of stationary equilibria. This is in contrast to the examples presented in this paper, which show that when the set of payoffs is convex and comprehensive, the kinks may well lead to asymptotic multiplicity of equilibria.

## 2. The bargaining game

Given a continuation probability  $\delta \in [0, 1)$ , define the  $n$ -player bargaining game  $\Gamma(\delta)$  as follows. The game starts in period zero. At the beginning of each period one of the players is randomly drawn to make a proposal. Player  $i$  is chosen with probability  $\mu_i > 0$ . The chosen player proposes an alternative  $x$  from the set  $V$  of feasible payoffs. All players (including the proposer) respond sequentially, with the sequence of responses being fixed throughout the game. Each responder can either accept or reject the current proposal. If all players accept the proposal, the game terminates and the proposal  $x$  is implemented, with player  $i$  receiving a payoff of  $x_i$ . Otherwise, the next period begins with probability  $\delta$ , whereas negotiations break down with probability  $1 - \delta$ .

**(A1)** The set  $V$  is a closed, convex, and proper subset of  $\mathbb{R}^n$  containing the point 0 in its interior.

**(A2)** The set  $V$  is comprehensive from below: If  $x \in V$  and  $y \in \mathbb{R}^n$  are such that  $y_i \leq x_i$  for each  $i$ , then  $y \in V$ .

**(A3)** The boundary  $\partial V$  of the set  $V$  is non-level: If  $x$  and  $y$  are points of  $\partial V$  such that  $x_i \leq y_i$  for each  $i = 1, \dots, n$ , then  $x = y$ .

We shall restrict our attention to pure stationary strategies by which we mean strategies such that (a) the proposal of any player is independent of the history of play and (b) the response of a player to a proposal only depends on the proposal itself. Thus a stationary strategy of player  $i$  consists of a proposal  $p_i \in V$  and an acceptance set  $A_i \subset V$ . Player  $i$  proposes the point  $p_i$  whenever player  $i$  is chosen to be a proposer and he accepts a proposal  $x$  if and only if  $x$  is an element of the set  $A_i$ . Theorems 1 and 2 below are well-known results.

**Theorem 1.** Suppose **(A1)–(A3)** are satisfied. Each subgame perfect equilibrium of the game  $\Gamma(\delta)$  in stationary strategies is a no-delay equilibrium. That is,  $p_i \in \cap_{j=1}^n A_j$  for each  $i = 1, \dots, n$ .

Since each equilibrium proposal is implemented without delay, the expected payoffs in a stationary subgame perfect equilibrium can be computed as  $r = \mu_1 p_1 + \dots + \mu_n p_n$ . Furthermore, the expected payoff to any player in any Nash equilibrium of  $\Gamma(\delta)$  is non-negative, since a player can attain a payoff of zero by rejecting all proposals.

**Theorem 2.** Suppose **(A1)–(A3)** are satisfied. Let  $p_1, \dots, p_n$  be the points of  $V$ . There exists a subgame perfect equilibrium of the game  $\Gamma(\delta)$  in stationary strategies with  $p_i$  the equilibrium proposal of player  $i$  if and only if each  $p_i$  is an element of  $\partial V$  and there exists a vector  $r$  such that for each  $i$  and  $j$

$$p_{ij} = \begin{cases} \frac{1 - \delta + \delta \mu_i}{\mu_i} r_i & \text{if } i = j \\ \delta r_j & \text{otherwise.} \end{cases}$$

The vector  $r$  in Theorem 2 coincides with the vector of expected payoffs.

We now turn to the asymptotic behavior of stationary subgame perfect equilibria as the continuation probability approaches one. Theorem 3 below shows that, under a differentiability assumption on the set  $V$ , the subgame perfect equilibria in stationary strategies are asymptotically unique: along any sequence of stationary subgame perfect equilibria as  $\delta$  tends to one, the equilibrium proposal of each player converges to the asymmetric Nash bargaining solution.

The asymmetric Nash product with weights  $\mu$  is the function  $\rho : V \rightarrow \mathbb{R}$  defined by

$$\rho(x) = \times_{i=1}^n (x_i)^{\mu_i}.$$

The point  $x$  of  $V$  is the asymmetric Nash bargaining solution with weights  $\mu$  ( $\mu$ -ANBS) if it maximizes the function  $\rho$  on the set  $V$ . Theorem 3 relies on the additional assumption of differentiability of the boundary of the set  $V$ .

**(A4)** The normal vector to  $V$  is unique up to a multiplier at any  $v \in \partial V$ . Equivalently, the set  $\partial V$  is a  $C^1$ -manifold.

**Theorem 3.** Suppose **(A1)–(A4)** are satisfied. Let  $\delta^k \in [0, 1)$  be a sequence of continuation probabilities converging to 1. For each  $k$  let  $(p_1^k, \dots, p_n^k)$  be the players' proposals in a stationary subgame perfect equilibrium of the game  $\Gamma(\delta^k)$ . As  $k$  goes to infinity each sequence  $p_i^k$  converges to the asymmetric Nash bargaining solution with weights  $\mu$ .

This result has been established by Hart and Mas-Colell (1996) in the case of equal recognition probabilities, Miyakawa (2008), Laruelle and Valenciano (2008), and Britz et al. (2010). We remark that when  $n = 2$  the conclusion of Theorem 3 is true under the assumptions **(A1)–(A3)**. For the two-player game where proposers are chosen with

equal probability, Binmore et al. (1986) show the convergence of equilibria to the symmetric Nash bargaining solution. There is no difficulty in extending this result to the case with general recognition probabilities  $\mu$ .

### 3. Two examples

We give two examples highlighting the role of the differentiability assumption (A4) for the asymptotic uniqueness of equilibria. Both examples satisfy Assumptions (A1)–(A3) but not (A4). The main feature of the example in Subsection 3.1 is that the game  $\Gamma(\delta)$  has a unique stationary subgame perfect equilibrium for each continuation probability, but the equilibria converge to a point different from the asymmetric Nash bargaining solution. A similar example, not satisfying (A3), has been developed independently by Kultti and Vartiainen (2010). The main feature of the example in Subsection 3.2 is the asymptotic multiplicity of equilibria: for each  $\delta$  the game  $\Gamma(\delta)$  has at least two distinct equilibria converging to two distinct limits as the continuation probability tends to one.

#### 3.1. Equilibria do not converge to ANBS

Consider a 3-player game with recognition probabilities  $\mu_i = 1/3$  for each  $i$ . The set  $V$  is the intersection of two halfspaces as depicted in Fig. 1:

$$V = H_1 \cap H_2,$$

where

$$H_1 = \{x \in \mathbb{R}^n \mid 2x_1 + x_2 + 3x_3 \leq 15\},$$

$$H_2 = \{x \in \mathbb{R}^n \mid x_1 + 2x_2 + 3x_3 \leq 15\}.$$

By Theorem 2 the equilibrium proposals are related to the expected payoffs  $r$  as depicted in Table 1.

The equilibrium proposal of each player lies in the boundary of the set  $V$ . A point  $x \in \mathbb{R}^n$  belongs to the boundary of the set  $V$  if and only if  $h(x) = 15$ , where the function  $h$  is defined by

$$h(x) = \max\{x_1 + 2x_2 + 3x_3, 2x_1 + x_2 + 3x_3\}$$

$$= x_1 + x_2 + 3x_3 + \max\{x_1, x_2\}.$$

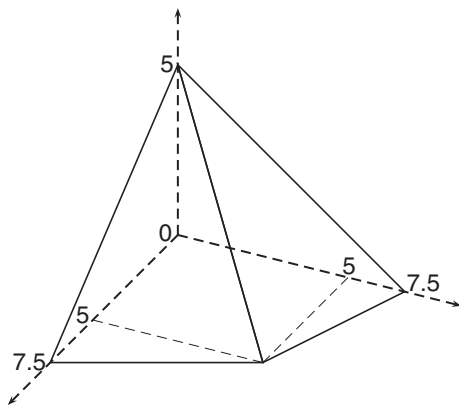


Fig. 1. The set  $V$  in the example of Subsection 3.1. The vertical axis represents the payoff to player 3.

Table 1  
Equilibrium proposals.

$p_1$	$(3-2\delta)r_1$	$\delta r_2$	$\delta r_3$
$p_2$	$\delta r_1$	$(3-2\delta)r_2$	$\delta r_3$
$p_3$	$\delta r_1$	$\delta r_2$	$(3-2\delta)r_3$

Therefore

$$15 = h(p_1) = (3-2\delta)r_1 + \delta r_2 + 3\delta r_3 + \max\{(3-2\delta)r_1, \delta r_2\}$$

$$15 = h(p_2) = \delta r_1 + (3-2\delta)r_2 + 3\delta r_3 + \max\{\delta r_1, (3-2\delta)r_2\}$$

$$15 = h(p_3) = \delta r_1 + \delta r_2 + 3(3-2\delta)r_3 + \max\{\delta r_1, \delta r_2\}.$$

Subtracting the first equation from the second we obtain

$$3(1-\delta)r_1 + \max\{(3-2\delta)r_1, \delta r_2\} = 3(1-\delta)r_2 + \max\{\delta r_1, (3-2\delta)r_2\}.$$

Now suppose  $r_1 < r_2$ . Then  $\max\{(3-2\delta)r_1, \delta r_2\} < (3-2\delta)r_2 \leq \max\{\delta r_1, (3-2\delta)r_2\}$ , implying that the left-hand side of the above equation is smaller than the right-hand side, a contradiction. Similarly, one shows that  $r_2 < r_1$  is impossible. We have thus established that  $r_1 = r_2$ . The system above reduces to the following:

$$15 = 3(2-\delta)r_1 + 3\delta r_3$$

$$15 = 3\delta r_1 + 3(3-2\delta)r_3.$$

Solving the system, we obtain the following values for the expected payoffs:

$$(r_1, r_2, r_3) = (15, 15, 10) / (6-\delta).$$

As  $\delta$  tends to one, the expected values converge to the vector  $(3, 3, 2)$ . It is clear that also the equilibrium proposal of each player converges to the point  $(3, 3, 2)$ . The asymmetric Nash bargaining solution can be easily computed to be  $(10/3, 10/3, 5/3)$ . Thus the unique equilibrium of the game  $\Gamma(\delta)$  fails to converge to the asymmetric Nash bargaining solution as  $\delta$  tends to one.

#### 3.2. Asymptotic multiplicity of equilibria

Consider a 3-player game with recognition probabilities  $\mu_i = 1/3$  for each  $i$ . The set of feasible payoffs, denoted by  $W$ , is the intersection of the set  $V$  as defined in the previous subsection and the halfspace  $H_3$ :

$$W = V \cap H_3,$$

where

$$H_3 = \{x \in \mathbb{R}^n \mid (2-5\varepsilon)x_1 + (2-5\varepsilon)x_2 + (3+4\varepsilon)x_3 \leq 3(2-5\varepsilon)(3+4\varepsilon)\},$$

and  $\varepsilon \in (0, 1/60)$ . We show that for  $\delta$  large enough the game  $\Gamma(\delta)$  has multiple equilibria. Moreover, one equilibrium converges to the point  $(3, 3, 2)$  and the other one to the point  $(3+4\varepsilon, 3+4\varepsilon, 2-5\varepsilon)$ .

##### 3.2.1. Equilibrium 1

We argue that the equilibrium of the game with set of utilities  $V = H_1 \cap H_2$  as considered in the preceding section remains an equilibrium in the game  $W$  for  $\delta$  large enough. In fact, using Theorem 2 we only have to verify that the players' proposals  $p_2$  and  $p_3$  in the bargaining equilibrium of the game  $V$  belong to the half-space  $H_3$ . To see this recall that the proposals  $p_1, p_2$  and  $p_3$  converge to the point  $v = (3, 3, 2)$ . The point  $v$  belongs to the interior of the half-space  $H_3$  provided that  $0 < \varepsilon < 1/60$ . But this implies that the proposals  $p_1, p_2$  and lie in the half-space  $H_3$  for  $\delta$  large enough, as desired.

##### 3.2.2. Equilibrium 2

Consider first the game where the set of feasible utilities is the entire half-space  $H_3$ . It follows from the results by Merlo and Wilson (1995) that such a game has a unique bargaining equilibrium, and that the equilibrium proposal of each player converges to the Nash bargaining solution of  $H_3$  as  $\delta$  tends to 1. The Nash bargaining solution of  $H_3$  is the point  $w = (3+4\varepsilon, 3+4\varepsilon, 2-5\varepsilon)$ .

Let  $\bar{p}_i$  be the equilibrium proposal of player  $i$  in the bargaining equilibrium of  $H_3$ . We argue that the bargaining equilibrium of the

game  $H_3$  is also an equilibrium of the game  $W$ . In fact, using Theorem 2 one only has to verify that the players' proposals  $\bar{p}_i$  all belong to the half-spaces  $H_1$  and  $H_2$ . The point  $w$  can be seen to be in the interior of both half-spaces  $H_1$  and  $H_2$  provided that  $\varepsilon > 0$ . It follows that for  $\delta$  large enough the players' proposals  $\bar{p}_i$  also belong to the half-spaces  $H_1$  and  $H_2$ , as desired.

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