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Equilibrium existence results for economies with price rigidities*

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Summary. An exchange economy with price rigidities and rationing is considered. The rationing systems allowed are very general. Several characterizations of the set of constrained equilibria are given, and new equilibrium existence results are provided. More specifically, well-known properties like the existence of equilibria without rationing of the numeraire commodity, and the existence of supply and demand constrained equilibria without rationing on the market of at least one commodity follow as special cases from the theorems proved. Finally it is shown that the equilibrium correspondence is upper semi-continuous, while it is continuous on a residual set of points. In order to prove these results a new continuity result for the budget correspondence is given.

1. Introduction

To prove the existence of a Walrasian equilibrium, prices have to be assumed to be completely flexible. If the constraints imposed by price rigidities are included in the set of admissible prices, a Walrasian equilibrium price does not necessarily exist. To reach an equilibrum of an economy with price rigidities, quantity adjustments by means of rationing are made instead of price adjustments. If price rigidities and quantity rationing are included into a general equilibrium model, an equilibrium concept has to be defined which differs from the one used by Debreu [6]. The corresponding equilibria are called constrained equilibria. In this paper the constraints on the set of admissible prices are given by lower and upper bounds on the prices on each market and are based on Drèze [9]. Making some substitutions it can be shown that models with prices depending on a general price index as studied in Dehez and Drèze [7] and van der Laan [15] are limit cases of the models considered in this paper. Other models with linked prices as studied in Kurz [12], van der Laan

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[15], Weddepohl [21], and Wu [22] cannot be considered as limit cases of the models in this paper. However, these authors consider different equilibrium concepts where in specified cases supply rationing on a market is allowed, while the price on this market could still be lowered.

The interpretation of the commodities in the model considered is the same as the one in Debreu [6]. There is not necessarily a commodity which serves as a numeraire commodity in the economy. On the other hand the existence of such a commodity is not excluded. In this case equilibria with rationing of the numeraire are considered too. The observation of demand rationing on the money market in Western economies provides some motivation for the study of these equilibria. An alternative modelling of price rigidities in an economy without a numeraire commodity is given in Nguyen and Whalley [17], [18]. Instead of using rationing to obtain an equilibrium they consider endogenously determined buying and selling prices which are determined by endogenously determined transaction costs.

It will be shown that the set of constrained equilibria is uncountable. This is in accordance with the examples given in Böhm and Müller [4] and with the results obtained in van der Laan and Talman [16]. A continuum of correspondences will be given such that each correspondence has a fixed point and each fixed point corresponds with a constrained equilibrium. Moreover, the set of correspondences can be chosen such that fixed points of different correspondences correspond to different equilibria and all constrained equilibria are obtained. In this way characterizations of the complete set of constrained equilibria are given.

A rationing system specifies the rationing schemes permitted in an economy. The rationing systems allowed in this paper are very general. It is shown that uniform rationing, rationing determined by initial endowments, rationing determined by market share, rationing determined by priority, and no restrictions on the rationing schemes are included as special cases. Without loss of generality the situation on a market, which is determined by the rationing scheme and the price, can be described by one parameter. This parameter determines either the level of demand rationing, or the price, or the level of supply rationing. It is shown in Section 3 that for every given value of this parameter on an a priori chosen market, a constrained equilibrium exists. Thereby a complete characterization of the equilibrium set is given.

In Kurz [12] and van der Laan [13] it has been remarked that supply rationing is observed more frequently in real world economies than demand rationing. In van der Laan [14] the existence of a constrained equilibrium with supply rationing only and with at least one market without rationing has been proved, using the technique of simplicial approximation of equilibria. These results will be generalized in Section 4. It is shown that there exists a constrained equilibrium with the parameter specifying the situation on a market less than or equal to a value specified for each market, and with equality on at least one market. Recent experiences in Eastern Europe show that also constrained equilibria with demand rationing are interesting. In Polterovich [19] models of Soviet-type economies are given where only rationing on the excess demand of consumers may occur. To include this case it is shown that there also exists a constrained equilibrium with the parameter specifying the situation on a market greater than or equal to a value specified for each market, and with equality on at least one market. A special case is the existence of a constrained equilibrium with only demand rationing and at least one market without rationing.

Given the consumption sets and the preferences of the consumers, the equilibrium correspondence assigns to each specification of initial endowments and set of admissible prices the set of all possible constrained equilibrium allocations. In Section 5 it is shown that the equilibrium correspondence is upper semi-continuous. The equilibrium correspondence is shown to be continuous on a residual set of points. The results concerning the equilibrium correspondence make clear that the set of constrained equilibria is stable against certain perturbations of the economy.

2. A model of an economy with price rigidities

In this section a model is described to deal with upper and lower bounds on the prices in an exchange economy defined by $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$. There are m consumers indexed $i = 1,\ldots,m$ and n commodities indexed $j = 1,\ldots,n$. Each consumer is defined by a consumption set X^i, a preference ordering $\succeq^i$ on X^i, and a vector of initial endowments w^i. The allocation of initial endowments is denoted by w, so $w = (w^1,\ldots,w^m)$. The set of admissible prices is denoted by $P_{(\underline{p},\bar{p})}$. In the following, for $k \in \mathbb{N}$, I_k denotes the set of integers $\{1,\ldots,k\}$ and $\mathbb{R}^k_+$ denotes the non-negative orthant of the k-dimensional Euclidean space $\mathbb{R}^k$. If $x, y \in \mathbb{R}^k$ then $x \geq y$ means $x_j \geq y_j$, $\forall j \in I_k$, $x > y$ means $x \geq y$ and $\exists j \in I_k$ such that $x_j > y_j$, and $x \gg y$ means $x_j > y_j$, $\forall j \in I_k$. The set $\{x \in \mathbb{R}^k \mid x \gg 0\}$ is denoted by $\mathbb{R}^k_{++}$. If S is a subset of $\mathbb{R}^k$ then Int(S) denotes the interior of S in $\mathbb{R}^k$. With respect to the economy E the following assumptions are made:

A1. X^i is a convex, closed, non-empty subset of $\mathbb{R}^n$, $X^i \subset \mathbb{R}^n_+$, and $X^i + \mathbb{R}^n_+ \subset X^i$.
A2. The preference ordering $\succeq^i$ on X^i is transitive, complete, continuous, weakly monotonic, and convex.
A3. The initial endowments w^i are an element of Int(X^i).
A4. The set of admissible prices is equal to $P_{(\underline{p},\bar{p})} = \{p \in \mathbb{R}^n_+ \mid \underline{p}_j \leq p_j \leq \bar{p}_j, \forall j \in I_n\}$, for given $\underline{p}, \bar{p} \in \mathbb{R}^n_{++}$ such that $\underline{p}_j \leq \bar{p}_j$, for every $j \in I_n$.

The requirement of weak monotonicity of the preference ordering is weaker than the assumption usually made in this stream of literature, where strong monotonicity with respect to some subset of commodities is made. The price set $P_{(\underline{p},\bar{p})}$ makes it possible to allow for a minimum (maximum) price for commodity $j \in I_n$, $\underline{p}_j$ ($\bar{p}_j$), to describe total inflexibility of the price ($\underline{p}_j = \bar{p}_j$), or to express a more moderate form of price rigidity ($\underline{p}_j < \bar{p}_j$). The sets of admissible prices allowed by Assumption A4 are slightly more general than the ones considered by Drèze [9] who gives results for cases with $\underline{p}_1 = \bar{p}_1 = 1$. An interesting special case of Assumption A4 is the one, also considered in the seminal work of Bènassy [2], where the price system is completely fixed, i.e., $\underline{p} = \bar{p}$. This case corresponds to the Keynesian point of view that in the short run prices are rigid and markets are cleared by means of quantity adjustments. Assumption A4 also allows for the interesting case that prices need not be completely inflexible in the short run, but that instead on each market prices may increase or decrease by some amount. Another point of view is that price rigidities are institutionally determined or are the result of strategic elements in the price setting process. Typical examples are minimum wages, minimum prices of agricultural products, maximum price controls to reduce inflation (see Cox [5]), price systems resulting from models with imperfect competition (see Bénassy [3]), price indexation, and the linkage between the wages of civil servants and the wages paid in industry.

Models with prices depending on a general price index are given in Dehez and Drèze [7] and van der Laan [15]. There the set of commodities I_n is partitioned in three sets H, I, and J, with H the set of free commodities, I the set of index commodities, and J the set of price following commodities. The price index function is a continuous function $\pi: \mathbb{R}^n_+ \to \mathbb{R}_+$ being homogeneous of degree one. The value of the price index is assumed to depend only on the prices of the index commodities. The prices of the price following commodities depend on the value of the price index. This results in the set $P_\pi = \{p \in \mathbb{R}^n_+ \mid \sum_{j \in H} p_j = 1,\ \underline{p}_j \pi(p) \leq p_j \leq \bar{p}_j \pi(p), \forall j \in I \cup J\}$ of admissible price systems. Under some assumptions, among which the one that $\underline{p}_j = \bar{p}_j, \forall j \in I$, it is shown in van der Laan [15] that the set P_π is equivalent to the set of admissible prices $P' = \{p \in \mathbb{R}^n_+ \mid p_j \geqslant 0,\ \forall j \in H,\ \underline{p}_j \leq p_j \leq \bar{p}_j,\ \forall j \in I \cup J\}$. The set P' is clearly a limit case of the sets of admissible prices considered in this paper.

In Kurz [12], Weddepohl [21], and Wu [22] models with rather general linkages between prices are considered. Partition the set of commodities in four groups, H, I, J, and K. For $j \in J \cup K$ let $\varphi_j: \mathbb{R}^n_+ \to \mathbb{R}_+$ and for $j \in K$ let $\psi_j: \mathbb{R}^n_+ \to \mathbb{R}_+$ be continuous index functions, which are homogeneous of degree one. Define the set of admissible prices $P_{(\varphi,\psi)} = \{p \in \mathbb{R}^n_+ \mid \varphi_j(p) \leq p_j,\ \forall j \in J,\ \varphi_j(p) \leq p_j \leq \psi_j(p),\ \forall j \in K,\ \sum_{j \in I_n} p_j = 1\}$. In Kurz [12] the case with $J = \emptyset$, $\varphi_j = \psi_j$, $\forall j \in K$, and the index functions only depending on prices in the set I is considered. In Wu [22] also the case with $K = \emptyset$ and the index functions only depending on prices in the set I is analyzed. Weddepohl [21] considers the set $P_{(\varphi,\psi)}$ making other assumptions with respect to the index functions. Van der Laan [15] also considers a special case corresponding to $J = \emptyset$ and for all $k \in K$, $\varphi_k(p) = \underline{p}_k \sum_{j \in I} p_j$, $\psi_k(p) = \bar{p}_k \sum_{j \in I} p_j$. Many different situations can be modelled by the set $P_{(\varphi,\psi)}$. The assumptions made by Kurz and Wu allow for a completely fixed price system. Weddepohl shows that the sets of admissible prices considered in Drèze [9] are not excluded by his assumptions. It is clear that it is also possible to model situations not corresponding to a set $P_{(\underline{p},\bar{p})}$ and it is therefore an interesting question whether the complete characterizations of constrained equilibria given in this paper can also be given for the models with a set of admissible prices $P_{(\varphi,\psi)}$. In this respect it should be remarked that in the papers concerning models with a set of admissible prices $P_{(\varphi,\psi)}$ weaker equilibrium concepts are used than in this paper. More specifically, it may happen in specified cases that excess supply on a market is rationed while the price on this market is not equal to its lower bound.

Since in an economy with price rigidities a Walrasian equilibrium does not necessarily exist, quantity adjustments instead of price adjustments are made to reach an equilibrium. A vector l^i such that $l^i \in -\mathbb{R}^n_+$ denotes a constraint on the excess supply of consumer $i \in I_m$ and a vector L^i such that $L^i \in \mathbb{R}^n_+$ denotes a constraint on the excess demand of consumer $i \in I_m$. The constrained budget set of consumer $i \in I_m$ at price $p \in \mathbb{R}^n_+$, rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$, and initial endowments $w^i \in \operatorname{Int}(X^i)$ is equal to $B^i(p, l^i, L^i, w^i) = \{x^i \in X^i \mid p \cdot x^i \leq p \cdot w^i, l^i \leq x^i - w^i \leq L^i\}$. The demand of consumer $i \in I_m$ at price $p \in \mathbb{R}^n_+$, rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$, and initial endowments $w^i\in\mathrm{Int}(X^i)$ is denoted by $\delta^i(p,l^i,L^i,w^i)$ and is given by $\delta^i(p,l^i,L^i,w^i)=\{x^i\in B^i(p,l^i,L^i,w^i)\mid x^i\succeq^i y^i,\ \forall y^i\in B^i(p,l^i,L^i,w^i)\}$. Consumer $i\in I_m$ is said to be constrained or rationed on his supply on market $k\in I_n$ if $\bar{x}^i\in\delta^i(p,\bar{l}^i,L^i,w^i)$ and $x^i\in\delta^i(p,l^i,L^i,w^i)$ implies $\bar{x}^i\succ^i x^i$, where $\bar{l}^i$ is the rationing scheme with $\bar{l}^i_j=l^i_j$, $\forall j\in I_n\setminus\{k\}$, and $\bar{l}^i_k=l^i_k-\varepsilon$ for some arbitrary positive real number ε. Using the convexity of preferences it can be shown that if consumer $i\in I_m$ is rationed on market k, then $x^i\in\delta^i(p,l^i,L^i,w^i)$ implies $l^i_k=x^i_k-w^i_k$. Similar remarks can be made with respect to demand rationing.

In order to show the upper semi-continuity of the demand correspondence it has to be shown that the constrained budget correspondence B^i is continuous for every $i\in I_m$. In Drèze [9], $w^i\in\mathrm{Int}(X^i)$ is considered as given and it is shown that the correspondence $B^i(\cdot,\cdot,\cdot,w^i)$ is continuous at $(p,l^i,L^i)\in\mathbb{R}^n_+\times-\mathbb{R}^n_+\times\mathbb{R}^n_+$ when, for some $j\in I_n$, $p_j>0$ and $l^i_j<0$. This last condition is restrictive if one wants to consider all constrained equilibria. The following example makes clear that even in case the initial endowments are in the interior of the consumption set and the consumption sets are compact, then the correspondence $B^i(\cdot,\cdot,\cdot,w^i)$ is not necessarily lower semi-continuous at a point $(p,l^i,L^i)\in\mathbb{R}^n_+\times-\mathbb{R}^n_+\times\mathbb{R}^n_+$ with $p\cdot l^i=0$.

Example 2.1. Let $X^i=\{x^i\in\mathbb{R}^2_+\mid x^i_1\le 2,\ x^i_2\le 2\}$, $w^i=(1,1)$, and for every $t\in\mathbb{N}$, $p^t=\left(\frac{1}{2},\frac{1}{t}\right)$, $l^{it}=\left(-\frac{1}{t},-\frac{1}{t}\right)$, and $L^{it}=(1,1)$. Then

$$B^i(p^t,l^{it},L^{it},w^i)=\left\{x^i\in X^i\mid 1-\frac{1}{t}\le x^i_1,\ 1-\frac{1}{t}\le x^i_2,\ \frac{1}{2}x^i_1+\frac{1}{t}x^i_2\le\frac{1}{2}+\frac{1}{t}\right\}.$$

Clearly, $(p^t,l^{it},L^{it})\to(p,l^i,L^i)$ with $p=(\frac{1}{2},0)$, $l^i=(0,0)$, and $L^i=(1,1)$, so $B^i(p,l^i,L^i,w^i)=\{x^i\in X^i\mid x^i_1=1,\ x^i_2\ge 1\}$. In particular $(1,2)\in B^i(p,l^i,L^i,w^i)$. However, for every $t\in\mathbb{N}$, $\max_{x^i\in B^i(p^t,l^{it},L^{it},w^i)}x^i_2=1\frac{1}{2}$, so $B^i(\cdot,\cdot,\cdot,w^i)$ is not lower semi-continuous at (p,l^i,L^i).

In Example 2.1 and in Drèze [9] some prices are allowed to be zero. This is excluded in this paper by Assumption A4. It turns out that if every price is positive, then the budget correspondence is continuous, even if $l^i_j=0$ for every $j\in I_n$. In order to show the upper semi-continuity of the equilibrium correspondence in Section 5, it is not enough to examine the continuity of the budget correspondence with initial endowments $w^i\in\mathrm{Int}(X^i)$ being fixed, but instead the more general case of Theorem 2.2 has to be considered. The proof of Theorem 2.2 is given in the Appendix.

Theorem 2.2. *Let X^i be closed, convex, $X^i\subset\mathbb{R}^n_+$, and $Int(X^i)\neq\emptyset$, $\forall i\in I_m$. Then the correspondence $B^i:\mathbb{R}^n_{++}\times-\mathbb{R}^n_+\times\mathbb{R}^n_+\times Int(X^i)$ into X^i is continuous.*

An equilibrium in an economy with price rigidities can be obtained by using quantity adjustments or rationing to clear the markets. The way rationing actually takes place may vary from market to market. It may be uniform for all consumers, depend on the amount of initial endowments owned by the various consumers, involve some priority system, etc. A rationing system describes all rationing schemes permitted in the economy. Let C^n be the set $\prod_{j=1}^n[0,1]$. Define $W=\prod_{i=1}^m\mathrm{Int}(X^i)$. The rationing system is specified by a pair of functions $(\tilde{l},\tilde{L})$ with $\tilde{l}:C^n\times W\to-\mathbb{R}^{mn}_+$

and $\tilde{L}: C^n \times W \to \mathbb{R}^{mn}_{+}$. It will not be excluded that the rationing system depends on the distribution of initial endowments $w \in W$. Given initial endowments $w \in W$, $\tilde{l}(C^n \times \{w\})$ denotes the set of all rationing schemes permitted on excess supply. Similarly, $\tilde{L}(C^n \times \{w\})$ is the set of all rationing schemes permitted on excess demand. Component $(i-1)n+j$ of $\tilde{l}$ is denoted by $\tilde{l}^i_j$, $\forall i \in I_m$, $\forall j \in I_n$. Moreover, $\tilde{l}^i = (\tilde{l}^i_1, \ldots, \tilde{l}^i_n)$ and $\tilde{l}_j = (\tilde{l}^1_j, \ldots, \tilde{l}^m_j)$. Given a rationing index $q \in C^n$ and initial endowments $w \in W$, $\tilde{l}^i_k(q, w)$ yields the constraint on the excess supply of consumer $i \in I_m$ on market $k \in I_n$. It will be assumed that $\tilde{l}_k(q, w)$ does not depend on q_j for $j \in I_n \backslash \{k\}$, hence the value of q_k determines the level of rationing on the excess supply on market k. Similar remarks can be made with respect to demand rationing. It will be assumed that the rationing system satisfies Assumption A5.

A5. The functions $\tilde{l}$ and $\tilde{L}$ are continuous and satisfy for every $q \in C^n$ and $w \in W$,

$$\begin{array}{ll}
\tilde{l}^i_j(q, w) = 0, \quad \text{if } q_j = 0, & \tilde{L}^i_j(q, w) > \sum_{h \in I_m \backslash \{i\}} w^h_j, \quad \text{if } q_j = 0, \\
\tilde{l}^i_j(q, w) < -w^i_j, \quad \text{if } q_j = 1, & \tilde{L}^i_j(q, w) = 0, \quad \text{if } q_j = 1, \\
\tilde{l}^i_j(q, w) = \tilde{l}^i_j(r, w), \quad \text{if } r \in C^n \text{ and } q_j = r_j, & \tilde{L}^i_j(q, w) = \tilde{L}^i_j(r, w), \quad \text{if } r \in C^n \text{ and } q_j = r_j.
\end{array}$$

This assumption is weaker than the one in Weddepohl [21] since it is not required that $\tilde{l}$ and $\tilde{L}$ are non-increasing. Below a few examples are given to illustrate the generality of the way the rationing system is described. Only the expressions for the rationing on excess supply are given. Similar expressions for the rationing on excess demand are easily obtained. It is not difficult to verify that the examples satisfy Assumption A5.

Example 2.3.

1. Uniform rationing as described in Drèze [9] is obtained by specifying $\tilde{l}^i_j(q, w) = -q_j \sum_{h=1}^m w^h_j$.
2. Let a real number $\varepsilon > 0$ be given. Defining $\tilde{l}^i_j(q, w) = -(1 + \varepsilon) q_j w^i_j$ gives rationing determined by initial endowments, equivalent to the rationing system in Kurz [12].
3. Rationing determined by market share as in Weddepohl [20]. Let numbers $\alpha^i_j > 0$ be given such that $\sum_{i=1}^m \alpha^i_j = 1$. Define $\tilde{l}^i_j(q, w) = -\dfrac{\alpha^i_j q_j \sum_{h=1}^m w^h_j}{\min\{\alpha^h_j \mid h \in I_m\}}$.
4. Rationing determined by priority (for a special case see Weddepohl [21]). Let $\sigma_j: I_m \to I_m$ be a permutation specifying the order in which consumers are rationed on their excess supply on market j. Define $\tilde{l}^i_j(q, w) = -\max\{\sigma_j(i) - m + mq_j, 0\} \sum_{h=1}^m w^h_j$.
5. Every element in the set $-\prod_{i=1}^m \prod_{j=1}^n [0, \sum_{h=1}^m w^h_j]$ is allowed as a rationing scheme on excess supply. Let f^1 be a continuous function from $[0, 1]$ onto a triangle. Such a function exists by a theorem of Peano about the existence of a space-filling curve (see Section 2.3 of Armstrong [1]). It is not difficult to construct a continuous function f^2 from the triangle onto C^2. Then the function $f^2 \circ f^1$ is a continuous function from $[0, 1]$ onto C^2. Using the function $f^2 \circ f^1$ it is not difficult to construct a continuous function g^2 from $[0, 1]$ onto C^2 having the additional property that $g^2(0) = (0, 0)$ and $g^2(1) = (1, 1)$. For $K \geq 3$ define $g^K: C^{K-1} \to C^K$ by $g^K(x_1, \ldots, x_{K-1}) = (g^{K-1}(x_1, \ldots, x_{K-2}), x_{K-1})$, $\forall x \in C^{K-1}$. Clearly g^K is continuous and onto for all $K \geq 2$. Let the function $h^2: [0, 1] \to C^2$ be equal to g^2. For $K \geq 3$ define

$h^K:[0,1]\rightarrow C^K$ by $h^K(x)=g^K(h^{K-1}(x))$, $\forall x\in[0,1]$. Then h^K is continuous and onto for all $K\geq 2$. Now a rationing system where every rationing scheme is allowed is obtained by taking $\tilde{l}^i_j(q,w)=-h^m_i(q_j)\sum_{h=1}^m w^h_j$.

From now on the definition of an exchange economy will be extended by a specification of a rationing system, so $E=(\{X^i,\succeq^i,w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l},\tilde{L}))$.

3. Existence of constrained equilibria

In the Sections 3 and 4 an exchange economy $E=(\{X^i,\succeq^i,w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l},\tilde{L}))$ satisfying Assumptions A1–A5 is given. Since the initial endowments will not vary in these two sections the argument w^i is suppressed in the notation of B^i, δ^i, $\tilde{l}$, and $\tilde{L}$. The following definition of a constrained equilibrium is closely related to the one given in Drèze [9].

Definition 3.1. (Constrained equilibrium) *A constrained equilibrium of the economy* $E=(\{X^i,\succeq^i,w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l},\tilde{L}))$ *is an element* $(x^{*1},\ldots,x^{*m},l^{*1},\ldots,l^{*m},L^{*1},\ldots,L^{*m},p^*)\in\prod_{i=1}^m X^i\times -\mathbb{R}^{mn}_+\times\mathbb{R}^{mn}_+\times P_{(\underline{p},\bar{p})}$ *satisfying*

1. $\forall i\in I_m: x^{*i}\in\delta^i(p^*,l^{*i},L^{*i})$;
2. $\sum_{i=1}^m x^{*i}-\sum_{i=1}^m w^i=0$;
3. $\forall j\in I_n: x^{*h}_j-w^h_j=L^{*h}_j$ *for some* $h\in I_m$ *implies* $x^{*i}_j-w^i_j>l^{*i}_j\forall i\in I_m$, *and* $x^{*h}_j-w^h_j=l^{*h}_j$ *for some* $h\in I_m$ *implies* $x^{*i}_j-w^i_j<L^{*i}_j\forall i\in I_m$;
4. $\forall j\in I_n: p^*_j<\bar{p}_j$ *implies* $L^{*i}_j>x^{*i}_j-w^i_j\forall i\in I_m$, *and* $p^*_j>\underline{p}_j$ *implies* $l^{*i}_j<x^{*i}_j-w^i_j\forall i\in I_m$;
5. $\exists q^{*1}\in C^n:\tilde{l}(q^{*1})=(l^{*1},\ldots,l^{*m})$ *and* $\exists q^{*2}\in C^n:\tilde{L}(q^{*2})=(L^{*1},\ldots,L^{*m})$.

By Conditions 1 and 2 the total demand of the consumers equals the total initial endowments available in the economy. Condition 3 implies that markets are frictionless, i.e., it does not occur that two consumers being on different sides of a market $j\in I_n$ are simultaneously rationed on this market. By Condition 4 there is no demand rationing on a market if the price is not equal to the upper bound on this market, and similarly supply rationing does not occur if the price on a market is greater than its lower bound. Finally, Condition 5 guarantees that the equilibrium rationing scheme is permitted in the economy.

Consider the state of a market $k\in I_n$ in a constrained equilibrium. By Conditions 3 and 4 there are three mutually exclusive possibilities. First, it may happen that rationing on the excess supply of at least one consumer $i\in I_m$ is binding. Then by Condition 3 there is no rationing on the excess demand of any consumer on market k. By Assumption A5 it therefore holds that $q^{*1}_k<1$ and, since non-binding rationing schemes are irrelevant, without loss of generality $q^{*2}_k=0$. Moreover, by Condition 4, $p^*_k=\underline{p}_k$. So the state of market k is in this case completely determined by the value of q^{*1}_k. The second possibility is that no consumer is rationed on market k. Then, without loss of generality, $q^{*1}_k=1$ and $q^{*2}_k=0$. Clearly, the price of commodity k is between $\underline{p}_k$ and $\bar{p}_k$, so $\underline{p}_k\leq p^*_k\leq\bar{p}_k$. The state of the market is completely determined by the value of p^*_k. Finally, the third possibility is that the excess demand of at least one consumer $h\in I_m$ is rationed on market k. Consequently, $L^{*h}_k=x^{*h}_k-w^h_k\leq\sum_{i=1}^m w^i_k-w^h_k$. By Assumption A5 this implies that $q^{*2}_k>0$ and

Condition 3 implies that without loss of generality $q_k^{*1} = 1$. By Condition 4, $p_k^* = \bar{p}_k$. The state of market k is completely determined by the value of q_k^{*2}.

The remarks in the paragraph above will be used to describe the state of a market $k \in I_n$ by one parameter $s_k \in [0, 1]$. If $0 \le s_k < \frac{1}{3}$ then the first possibility above occurs with price $p_k = \underline{p}_k$ and rationing scheme $(l_k^1, \ldots, l_k^m, L_k^1, \ldots, L_k^m) = (\tilde{l}_k(q^1), \tilde{L}_k(q^2))$ when we take $q_k^1 = 3s_k$ and $q_k^2 = 0$. If $\frac{1}{3} \le s_k \le \frac{2}{3}$ then we are in the second case with price $p_k = \underline{p}_k(2 - 3s_k) + \bar{p}_k(3s_k - 1)$ and rationing scheme $(l_k^1, \ldots, l_k^m, L_k^1, \ldots, L_k^m) = (\tilde{l}_k(q^1), \tilde{L}_k(q^2))$, $q_k^1 = 1$ and $q_k^2 = 0$. If $\frac{2}{3} < s_k \le 1$ then the third case occurs with price $p_k = \bar{p}_k$ and rationing scheme $(l_k^1, \ldots, l_k^m, L_k^1, \ldots, L_k^m) = (\tilde{l}_k(q^1), \tilde{L}_k(q^2))$ when $q_k^1 = 1$ and $q_k^2 = 3s_k - 2$. Therefore define the functions $\hat{p}: C^n \to \mathbb{R}^n_{++}$, $\hat{l}: C^n \to -\mathbb{R}^{mn}_+$, $\hat{L}: C^n \to \mathbb{R}^{mn}_+$, and the correspondence $\hat{\delta}^i: C^n \to \mathbb{R}^n$, $\forall i \in I_m$, by

$$\begin{aligned}
&\hat{p}_j(s) = \max\{\underline{p}_j, \min\{\underline{p}_j(2 - 3s_j) + \bar{p}_j(3s_j - 1), \bar{p}_j\}\}, \quad \forall s \in C^n, \quad \forall j \in I_n,\\
&\hat{l}(s) = \tilde{l}(\min\{\imath^n, 3s\}), \quad \forall s \in C^n,\\
&\hat{L}(s) = \tilde{L}(\max\{0, 3s - 2\imath^n\}), \quad \forall s \in C^n,\\
&\hat{\delta}^i(s) = \delta^i(\hat{p}(s), \hat{l}(s), \hat{L}(s)), \quad \forall s \in C^n,
\end{aligned}$$

where $\imath^n$ denotes the n-dimensional vector containing only ones.

Theorem 3.2. *Let an economy* $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfying Assumptions A1–A5 be given. If for some* $s^* \in C^n$ *there exists* $x^{*i} \in \hat{\delta}^i(s^*)$, $\forall i \in I_m$, *such that* $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$, *then* $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ *is a constrained equilibrium of E.*

Proof. By the definitions of $\hat{p}$, $\hat{l}$, and $\hat{L}$, $\hat{p}(s^*) \in P_{(\underline{p},\bar{p})}$. There are $q^{*1}, q^{*2} \in C^n$ such that $\hat{l}(s^*) = \tilde{l}(q^{*1})$ and $\hat{L}(s^*) = \tilde{L}(q^{*2})$. Therefore Condition 5 of Definition 3.1 is satisfied. Conditions 1 and 2 of Definition 2.1 of a constrained equilibrium, requiring that $x^{*i} \in \delta^i(\hat{p}(s^*), \hat{l}^i(s^*), \hat{L}^i(s^*)) = \hat{\delta}^i(s^*)$ and $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$ are clearly satisfied. Now the Conditions 3 and 4 of Definition 2.1 are examined. For every $i \in I_m$ and $j \in I_n$, $x_j^{*i} - w_j^i = \sum_{h=1}^m w_j^h - \sum_{h \in I_m \setminus \{i\}} x_j^{*h} - w_j^i \le \sum_{h \in I_m \setminus \{i\}} w_j^h$.

If for some $k \in I_n$ there is an $h \in I_m$ such that $x_k^{*h} - w_k^h = \hat{L}_k^h(s^*)$, then $\hat{L}_k^h(s^*) \le \sum_{i \in I_m \setminus \{h\}} w_k^i$, therefore $s_k^* > \frac{2}{3}$ and hence $\hat{l}_k^i(s^*) < -w_k^i \le x_k^{*i} - w_k^i$, $\forall i \in I_m$, so the first part of Condition 3 is satisfied.

If for some $k \in I_n$ there is an $h \in I_m$ such that $x_k^{*h} - w_k^h = \hat{l}_k^h(s^*)$, then $\hat{l}_k^h(s^*) \ge -w_k^h$, therefore $s_k^* < \frac{1}{3}$ and hence $\hat{L}_k^i(s^*) > \sum_{h \in I_m \setminus \{i\}} w_k^h \ge x_k^{*i} - w_k^i$, $\forall i \in I_m$, so the second part of Condition 3 is satisfied.

If for some $k \in I_n$, $\hat{p}_k(s^*) < \bar{p}_k$, then $s_k^* < \frac{2}{3}$ and therefore $\hat{L}_k^i(s^*) > \sum_{h \in I_m \setminus \{i\}} w_k^h \ge x_k^{*i} - w_k^i$, $\forall i \in I_m$. Also, if for some $k \in I_n$, $\hat{p}_k(s^*) > \underline{p}_k$, then $s_k^* > \frac{1}{3}$ and therefore $\hat{l}_k^i(s^*) < -w_k^i \le x_k^{*i} - w_k^i$, $\forall i \in I_m$. So Condition 4 is also satisfied. Q.E.D.

Theorem 3.2 gives an easy characterization of equilibria by using the functions $\hat{p}$, $\hat{l}^i$, and $\hat{L}^i$. The following theorem states that if these functions are used, all possible constrained equilibrium allocations, prices, and binding rationing schemes are indeed obtained. The proof of Theorem 3.3 is straightforward and therefore omitted.

Theorem 3.3. *Let an economy* $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfying Assumptions A1–A5 be given. If* $(x^{*1}, \ldots, x^{*m}, l^{*1}, \ldots, l^{*m}, L^{*1}, \ldots, L^{*m}, p^*)$ *is a constrained equilibrium of E, then there exists an* $s^* \in C^n$ *such that* $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ *is* *a constrained equilibrium of E. Moreover,* s^* *can be chosen such that* $x_j^{*h} - w_j^h = l_j^{*h}$ *for some* $h \in I_m$ *implies* $\hat{l}_j^i(s^*) = l_j^{*i}$, $\forall i \in I_m$, $x_j^{*h} - w_j^h = L_j^{*h}$ *for some* $h \in I_m$ *implies* $\hat{L}_j^i(s^*) = L_j^{*i}$, $\forall i \in I_m$, *and* $\hat{p}(s^*) = p^*$.

It is easily verified that $s^* = 0$ induces a constrained equilibrium with complete supply rationing on every market and, similarly, $s^* = \iota^n$ induces a constrained equilibrium with complete demand rationing on every market. These two equilibria are called the trivial equilibria. Consequently, there exists a constrained equilibrium where the state of a given market $k \in I_n$ is 0 or 1. In Theorem 3.4 it is shown that given any state $\sigma \in [0,1]$ of a market $k \in I_n$ there is a corresponding constrained equilibrium.

Theorem 3.4. *Let an economy* $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfying Assumptions A1–A5 be given. Let* $k \in I_n$ *and* $\sigma \in [0,1]$ *be given. Then there exists an* $s^* \in C^n$ *such that* $s_k^* = \sigma$ *and* s^* *induces a constrained equilibrium* $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ *of E.*

Proof. For every $i \in I_m$, define the set $\bar{X}^i$ by $\bar{X}^i = \{x^i \in X^i \mid \underline{p}_j x_j^i \leq \bar{p} \cdot w^i, \forall j \in I_n\}$, and define $C_{k,\sigma}^n = \{s \in C^n \mid s_k = \sigma\}$. Notice that $x^i \in \hat{\delta}^i(s)$ implies $x^i \in \bar{X}^i$. Define the correspondence $\mu_{k,\sigma}: \prod_{i=1}^m \bar{X}^i \rightarrow C_{k,\sigma}^n$ by $\mu_{k,\sigma}(x^1, \ldots, x^m) = \{\bar{s} \in C_{k,\sigma}^n \mid \bar{s} \cdot \sum_{i=1}^m (x^i - w^i) \geq s \cdot \sum_{i=1}^m (x^i - w^i), \forall s \in C_{k,\sigma}^n\}$, $\forall (x^1, \ldots, x^m) \in \prod_{i=1}^m \bar{X}^i$. Obviously, if $j \in I_n \setminus \{k\}$, $\sum_{i=1}^m (x_j^i - w_j^i) > 0$, and $s \in \mu_{k,\sigma}(x^1, \ldots, x^m)$, then $s_j = 1$. Similarly, $j \in I_n \setminus \{k\}$, $\sum_{i=1}^m (x_j^i - w_j^i) < 0$, and $s \in \mu_{k,\sigma}(x^1, \ldots, x^m)$ implies $s_j = 0$. Since only weak monotonicity of preference orderings is assumed, Walras' law does not necessarily hold. For every $i \in I_m$ define the correspondence $\Omega^i: \mathbb{R}_{++}^n \rightarrow \bar{X}^i$ by $\Omega^i(p) = \{x^i \in \bar{X}^i \mid p \cdot x^i = p \cdot w^i\}$, $\forall p \in \mathbb{R}_{++}^n$, define the correspondence $\bar{\delta}^i: C_{k,\sigma}^n \rightarrow \bar{X}^i$ by $\bar{\delta}^i(s) = \hat{\delta}^i(s) \cap \Omega^i(\hat{p}(s))$, $\forall s \in C_{k,\sigma}^n$, and define the correspondence $\varphi: \prod_{i=1}^m \bar{X}^i \times C_{k,\sigma}^n \rightarrow \prod_{i=1}^m \bar{X}^i \times C_{k,\sigma}^n$ by

$$\varphi(x^1, \ldots, x^m, s) = \prod_{i=1}^m \bar{\delta}^i(s) \times \mu_{k,\sigma}(x^1, \ldots, x^m), \quad \forall (x^1, \ldots, x^m, s) \in \prod_{i=1}^m \bar{X}^i \times C_{k,\sigma}^n.$$

It follows easily from the maximum theorem that $\mu_{k,\sigma}: \prod_{i=1}^m \bar{X}^i \rightarrow C_{k,\sigma}^n$ is non-empty valued and upper semi-continuous on $\prod_{i=1}^m \bar{X}^i$. By Theorem 2.2, the continuity of the functions $\hat{p}, \hat{l}, \hat{L}$, and by the maximum theorem it follows that the correspondence $\hat{\delta}^i$ is non-empty valued and upper semi-continuous on C^n. It is not difficult to show that the correspondence $\Omega^i \circ \hat{p}$ is non-empty valued and upper semi-continuous on C^n. Since for every $s \in C^n$, $\hat{\delta}^i(s) \cap \Omega^i(\hat{p}(s)) \neq \emptyset$ the correspondence $\bar{\delta}^i$ is non-empty valued and upper semi-continuous on $C_{k,\sigma}^n$ by Hildenbrand [11] (B.III Proposition 2). It is easily verified that the correspondence φ is convex-valued. Since $\prod_{i=1}^m \bar{X}^i \times C_{k,\sigma}^n$ is a compact, convex, and non-empty set all conditions of Kakutani's fixed point theorem are satisfied. So the correspondence φ has a fixed point $(x^{*1}, \ldots, x^{*m}, s^*) \in \prod_{i=1}^m \bar{X}^i \times C_{k,\sigma}^n$ satisfying $x^{*i} \in \bar{\delta}^i(s^*) = \hat{\delta}^i(s^*) \cap \Omega^i(\hat{p}(s^*))$, $\forall i \in I_m$, and $s^* \in \mu_{k,\sigma}(x^{*1}, \ldots, x^{*m})$. It will be shown that $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ is a constrained equilibrium of E. Using Theorem 3.2 it is sufficient to show that $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$.

Suppose there exists a $j' \in I_n$ such that $\sum_{i=1}^m x_{j'}^{*i} < \sum_{i=1}^m w_{j'}^i$. Two cases are possible, $j' \neq k$ and $j' = k$. If $j' \neq k$ then by the definition of $\mu_{k,\sigma}$, $s_{j'}^* = 0$. Consequently,

$\sum_{i=1}^{m} x_{j'}^{*i} \geq \sum_{i=1}^{m} (w_{j'}^{i} + \hat{l}_{j'}^{i}(s^*)) = \sum_{i=1}^{m} w_{j'}^{i}$, a contradiction. If $j' = k$ then there is a $j'' \in I_n \setminus \{k\}$ such that $\sum_{i=1}^{m} x_{j''}^{*i} > \sum_{i=1}^{m} w_{j''}^{i}$. Using the definition of $\mu_{k,\sigma}$ this implies that $s_{j''}^{*} = 1$. Therefore, $\sum_{i=1}^{m} x_{j''}^{*i} \leq \sum_{i=1}^{m} (w_{j''}^{i} + \hat{L}_{j''}^{i}(s^*)) = \sum_{i=1}^{m} w_{j''}^{i}$, a contradiction.

Consequently, $\sum_{i=1}^{m} x^{*i} \geq \sum_{i=1}^{m} w^{i}$, and since $\hat{p}(s^*) \cdot \sum_{i=1}^{m} (x^{*i} - w^{i}) = 0$ it follows that $\sum_{i=1}^{m} x^{*i} = \sum_{i=1}^{m} w^{i}$. Q.E.D.

Using stronger versions of Assumptions A1 and A2 a similar result is obtained in van der Laan and Talman [16] for the case of the uniform rationing system. Notice that if $\sigma \in (0, \frac{1}{3})$ and the rationing system is not the uniform one, then it cannot be guaranteed that for every $i \in I_m$ there exists some $j \in I_n$ such that $\hat{l}_j^i(s^*) < 0$. Therefore the proof of Drèze [9] of the continuity of B^i cannot be used to prove the existence of a constrained equilibrium in this case and we need Theorem 2.2.

Suppose that one of the commodities is a numeraire commodity, say commodity 1. A constrained equilibrium without rationing of the numeraire commodity is called a Drèze equilibrium. The existence of such a Drèze equilibrium follows immediately from Theorem 3.4 by taking $k = 1$ and $\sigma \in [\frac{1}{3}, \frac{2}{3}]$.

Corollary 3.5. *Let an economy* $E = (\{X^i, \succeq^i, w^i\}_{i=1}^{m}, P_{(\underline{p}, \bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfying Assumptions A1–A5 be given. Let one of the commodities be a numeraire commodity. Then there exists a Drèze equilibrium of E.*

If follows immediately that there exists a constrained equilibrium for every state of the market of the numeraire commodity. It is also interesting to consider for example the labour market. Then Theorem 3.4 makes clear that every state of the labour market can be sustained as a constrained equilibrium. By Theorem 3.3 the set of all constrained equilibria is obtained by considering the constrained equilibria corresponding with every possible state $\sigma \in [0, 1]$ of some given market $k \in I_n$. Hence Theorem 3.4 gives a complete classification of all constrained equilibria and, moreover, Theorem 3.4 makes clear that there are uncountably many constrained equilibria. Since there are so many constrained equilibria one might argue that the concept is not well-defined. On the other hand one might argue that the fact that there are many constrained equilibria necessitates a dynamic study like a study of adjustment processes, specifying which equilibrium will result given the initial state of the economy. Since in the trivial equilibria every consumer keeps his initial endowments, this equilibrium will usually be Pareto dominated by other constrained equilibria. Therefore one might expect that not one of the trivial equilibria will result. Since it is not difficult to construct examples of economies with a numeraire commodity such that every Drèze equilibrium is Pareto dominated by another constrained equilibrium, it is also not clear that a Drèze equilibrium will result.

In Assumption A4 the case where the price is zero on some market $j \in I_n$ is excluded. Nevertheless the existence of a Drèze equilibrium for such an economy with a numeraire commodity is shown in Drèze [9]. It is therefore interesting to consider the question whether the result of Theorem 3.4 can be obtained in this case too. In Example 2.1 it is shown that some technical difficulties arise. Example 3.6 makes clear that indeed strictly positive prices are necessary for the result of Theorem 3.4.

Example 3.6. Consider an economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l}, \tilde{L}))$ with $\underline{p}_k = 0$ for some market $k \in I_n$. Assume that consumer 1 has strictly monotonic preferences with respect to commodity k. Hence $x^1 \in \delta^1(p, l^1, L^1)$ implies $x_k^1 - w_k^1 = L_k^1$. Suppose $s^* \in C^n$ induces a constrained equilibrium $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ of E. Suppose $s_k^* \leq \frac{1}{3}$. Then $\hat{p}_k(s^*) = 0$, $\hat{L}_k^1(s^*) > \sum_{h \in I_m \setminus \{1\}} w_k^h$ and therefore $\sum_{i=1}^m x_k^{*i} \geq x_k^{*1} = w_k^1 + \hat{L}_k^1(s^*) > \sum_{i=1}^m w_k^i$, a contradiction. Hence there are no constrained equilibria where the state of market k, s_k^*, lies in the interval $[0, \frac{1}{3}]$. So there are no constrained equilibria with binding supply rationing on market k. Assume that also $\bar{p}_k = 0$. Then it can be shown in a similar way that if a constrained equilibrium is induced by some $s^* \in C^n$, then $s_k^* > \frac{2}{3}$. If the preferences of every consumer are strictly monotone with respect to commodity k, then it holds for every constrained equilibrium that the demand of commodity k is completely rationed, i.e., s^* may induce a constrained equilibrium only if $\hat{L}_k^i(s^*) = 0$, $\forall i \in I_m$.

4. Existence of extended supply and demand constrained equilibria

In Kurz [12] and van der Laan [13] it has been remarked that in Western economies supply rationing occurs more frequently than demand rationing. Examples are the rationing of labour supplied, resulting in unemployment, and quotas on the supply of agricultural products. One of the reasons given in van der Laan [13] for this phenomenon is that constraints on the supply side can often more easily be realized than constraints on the demand side since the number of sellers is usually less than the number of buyers. Although on the labour market the reverse is true, rationing the supply of labour is easily realized by restricting the number of hours worked. A supply constrained equilibrium is a constrained equilibrium without rationing on excess demand, while on one market there is no rationing at all. The commodity of the latter market can be chosen ex post as a numeraire commodity which is not rationed. The existence of a supply constrained equilibrium is shown in van der Laan [14].

In case in equilibrium only supply rationing is allowed, this can be modelled by requiring that $s^* \in C^n$ induces a constrained equilibrium $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ satisfying the conditions of Definition 3.1 and moreover $s_j^* \leq \frac{2}{3}$, $\forall j \in I_n$. If it should hold that there is at least one market without rationing, then this can be modelled by requiring in addition that for some market $k \in I_n$, $s_k^* \geq \frac{1}{3}$. Notice that the existence of a constrained equilibrium induced by s^* with $s_k^* = \sigma$ for $k \in I_n$ and $\sigma \in [0, 1]$ shown in Theorem 3.4 does not show the existence of a supply constrained equilibrium, since only the state of market k is considered. It is possible to give a generalization of the concept of a supply constrained equilibrium. Let a vector $\alpha \in C^n$ be given. A constrained equilibrium induced by $s \in C^n$ where for each market $j \in I_n$, s_j is less than or equal to α_j while on at least one market $k \in I_n$, s_k is equal to α_k is called an extended supply constrained equilibrium with respect to α (α-SCE). In this way it can be modelled for example that on some markets a given amount of rationing on excess demand is allowed, or that on some markets some rationing on excess supply is always present, an example is the existence of a natural rate of unemployment. The existence of an α-SCE for an arbitrary $\alpha \in C^n$ is shown in the following theorem.

Theorem 4.1. *Let an economy* $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(p,\bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfying Assumptions A1–A5 be given. Then for every* $\alpha \in C^n$ *there exists an* α*-SCE* $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ *of the economy E, i.e.,* $s^* \in C^n$ *satisfies* $s^* \leq \alpha$ *and* $s_k^* = \alpha_k$ *for some* $k \in I_n$.

Proof. Define the set $S^n = \{s \in \mathbb{R}_+^n \mid \sum_{j=1}^n s_j = 1\}$ and define $\bar{X}^i$, $\forall i \in I_m$, as in the proof of Theorem 3.4. Define the correspondence μ from $\prod_{i=1}^m \bar{X}^i$ into S^n by $\mu(x^1, \ldots, x^m) = \{\bar{s} \in S^n \mid \bar{s} \cdot \sum_{i=1}^m (x^i - w^i) \geq s \cdot \sum_{i=1}^m (x^i - w^i),\ \forall s \in S^n\}$, $\forall (x^1, \ldots, x^m) \in \prod_{i=1}^m \bar{X}^i$. Using the maximum theorem it follows immediately that the correspondence μ from $\prod_{i=1}^m \bar{X}^i$ into S^n is non-empty valued and upper semi-continuous. It is also easily verified that if $(x^1, \ldots, x^m) \in \prod_{i=1}^m \bar{X}^i$ and for some j, $j' \in I_n$ it holds that $\sum_{i=1}^m x_{j'}^i - \sum_{i=1}^m w_{j'}^i > \sum_{i=1}^m x_j^i - \sum_{i=1}^m w_j^i$, then $s_j = 0$ for every $s \in \mu(x^1, \ldots, x^m)$. Define component $j \in I_n$ of the function $f: S^n \to C^n$ by

$$f_j(s) = \frac{\alpha_j s_j}{\max\{s_1, \ldots, s_n\}}, \quad \forall s \in S^n.$$

Define the correspondence $\tilde{\delta}^i: S^n \to \bar{X}^i$ by $\tilde{\delta}^i(s) = \bar{\delta}^i(f(s))$, where $\bar{\delta}^i$ is as defined in the proof of Theorem 3.4. Since f is continuous on S^n and $\bar{\delta}^i$ is upper semi-continuous on C^n it follows that $\tilde{\delta}^i$ is upper semi-continuous on S^n. Define the correspondence $\varphi: \prod_{i=1}^m \bar{X}^i \times S^n \to \prod_{i=1}^m \bar{X}^i \times S^n$ by

$$\varphi(x^1, \ldots, x^m, s) = \prod_{i=1}^m \tilde{\delta}^i(s) \times \mu(x^1, \ldots, x^m), \quad \forall (x^1, \ldots, x^m, s) \in \prod_{i=1}^m \bar{X}^i \times S^n.$$

It is easily verified that φ satisfies the conditions of Kakutani's fixed point theorem. Let $(x^{*1}, \ldots, x^{*m}, s^*)$ be a fixed point of φ. Clearly $x^{*i} \in \tilde{\delta}^i(s^*) = \hat{\delta}^i(f(s^*)) \cap \Omega^i(\hat{p}(f(s^*)))$, $\forall i \in I_m$, and $s^* \in \mu(x^{*1}, \ldots, x^{*m})$. By definition of the function f, $f(s^*) \leq \alpha$ and $f_k(s^*) = \alpha_k$ for some $k \in I_n$. Using Theorem 3.2 it follows therefore that $(x^{*1}, \ldots, x^{*m}, \hat{l}(f(s^*)), \hat{L}(f(s^*)), \hat{p}(f(s^*)))$ is an α-SCE if it can be shown that $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$.

Suppose there is some $j' \in I_n$ such that $\sum_{i=1}^m x_{j'}^{*i} - \sum_{i=1}^m w_{j'}^i < 0$. Then there is a $j'' \in I_n$ such that $\sum_{i=1}^m x_{j''}^{*i} - \sum_{i=1}^m w_{j''}^i > 0$. Using the properties of μ this implies that $s_{j'}^* = 0$ and therefore $f_{j'}(s^*) = 0$. So $\sum_{i=1}^m x_{j'}^{*i} - \sum_{i=1}^m w_{j'}^i \geq \sum_{i=1}^m \hat{l}_{j'}^i(f(s^*)) = 0$, a contradiction. Consequently, $\sum_{i=1}^m x^{*i} - \sum_{i=1}^m w^i \geq 0$ and this immediately yields $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$. Q.E.D.

Notice that if $\min\{\alpha_j \mid j \in I_n\} < \frac{1}{3}$ it cannot be guaranteed that for every $i \in I_m$ there exists some $j \in I_n$ such that $\hat{l}_j^i(s) < 0$, $\forall s \in f(S^n)$, with f as defined in the proof of Theorem 4.1. Therefore the original proof of Drèze [9] of the continuity of the budget correspondence cannot be used to prove the existence of an α-SCE if $\min\{\alpha_j \mid j \in I_n\} < \frac{1}{3}$ and again the result of Theorem 2.2 is needed.

Since a supply constrained equilibrium is defined as a constrained equilibrium such that there is no market with demand rationing and there is at least one market without supply rationing, the existence of a supply constrained equilibrium follows immediately from Theorem 4.1 by taking for example $\alpha = \frac{1}{2}\iota^n$.

Corollary 4.2. *Let an economy* $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(p,\bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfying Assumptions A1–A5 be given. Then there exists a supply constrained equilibrium of E.*

In Example 4.3 it is shown that strictly positive prices are necessary for the result of Theorem 4.1.

Example 4.3. Consider an economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l}, \tilde{L}))$ with $\underline{p}_k = 0$ for some market $k \in I_n$ and assume that consumer 1 has strictly monotonic preferences with respect to commodity k. In Example 3.6 it has been shown that s^* may induce a constrained equilibrium only if $s_k^* > \frac{1}{3}$. Hence an α-SCE does not exist if $\alpha_k \leq \frac{1}{3}$. Moreover, if $\bar{p}_k = 0$ then, following the reasoning of Example 3.6, s^* may induce a constrained equilibrium only if $s^* > \frac{2}{3}$. This implies that an α-SCE does not exist if $\alpha_k \leq \frac{2}{3}$ and that a supply constrained equilibrium does not exist in this case.

The requirement that for at least one market $k \in I_n$ the state is given by α_k makes clear that α^1 and α^2 will typically induce different constrained equilibria if $\alpha^1 \ll \alpha^2$. The set of all constrained equilibria is obtained by considering for example the $\lambda\iota^n$-SCE for every $\lambda \in [0, 1]$. This gives another complete classification of all constrained equilibria.

Recent experiences in Eastern Europe and the former Soviet Union make clear that demand constrained equilibria and extended demand constrained equilibria are interesting too. In Polterovich [19] some general equilibrium type models of centrally planned economies are considered. In none of these models rationing on excess supply exists, while rationing on excess demand might be binding on all markets. In our set-up a demand constrained equilibrium is a constrained equilibrium without rationing on excess supply, while on one market there is no demand rationing. It is possible to generalize the concept of a demand constrained equilibrium. Let a vector $\beta \in C^n$ be given. A constrained equilibrium where on each market $j \in I_n$, s_j is greater than or equal to β_j while on at least one market $k \in I_n$, s_k is equal to β_k is called an extended demand constrained equilibrium with respect to the vector β (β-DCE). The existence of a β-DCE for an arbitrary vector $\beta \in C^n$ is stated in the following theorem, which can be proved in a way similar to Theorem 4.1.

Theorem 4.4. *Let an economy* $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfying Assumptions A1–A5 be given. Then for every* $\beta \in C^n$ *there exists a* β*-DCE* $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ *of the economy E, i.e.,* $s^* \in C^n$ *satisfies* $s^* \geq \beta$ *and for some* $k \in I_n$, $s_k^* = \beta_k$.

The existence of a demand constrained equilibrium follows as a corollary of Theorem 4.4 by taking for example $\beta = \frac{1}{2}\iota^n$. It should be noticed that all possible constrained equilibria are obtained by the correspondences used in the Theorems 3.4, 4.1, and 4.4, whose fixed points are constrained equilibria. By Theorem 3.3 every constrained equilibrium for an economy E can be written as $(x^{*1}, \ldots, x^{*m}, \hat{l}(s^*), \hat{L}(s^*), \hat{p}(s^*))$ for some $s^* \in C^n$. Such a constrained equilibrium satisfies the conditions of Theorem 3.4 for every $k \in I_n$ and σ equal to s_k^*. It satisfies the conditions of Theorem 4.1 for every $\alpha \in C^n$ satisfying $\alpha \geq s^*$ and for some $k \in I_n$, $\alpha_k = s_k^*$. It also satisfies the conditions of Theorem 4.4 for every $\beta \in C^n$ satisfying $\beta \leq s^*$ and for some $k \in I_n$, $\beta_k = s_k^*$.

5. The equilibrium correspondence

Let a consumption set X^i and a preference relation $\succeq^i$ be given for every consumer $i \in I_m$. Moreover, suppose the rationing system $(\tilde{l}, \tilde{L})$ is given. In this section it is shown that the equilibrium correspondence, which assigns to every specification of initial endowments and set of admissible prices the set of all constrained equilibrium allocations, is upper semi-continuous. Moreover, it can be shown that it is continuous on a residual subset of the domain. These are interesting properties since they imply that the set of equilibrium allocations is stable against perturbations in the initial endowments or the set of admissible prices. Since in this section the initial endowments may vary they are now included in the notation of $B^i, \delta^i, \tilde{l}$, and $\tilde{L}$. Define the set $R = \{r \in \mathbb{R}^{2n}_{++} \mid r_j \leq r_{j+n}, \forall j \in I_n\}$. Notice that an element $r \in R$ can be used to specify the set of admissible prices P_r satisfying Assumption A4. Functions $\hat{p}: C^n \times R \to \mathbb{R}^n_{++}$, $\hat{l}: C^n \times W \to -\mathbb{R}^{mn}_+$, and $\hat{L}: C^n \times W \to \mathbb{R}^{mn}_+$ defined by

$$\begin{aligned} &\hat{p}_j(s,r) = \max\{r_j, \min\{r_j(2-3s_j) + r_{j+n}(3s_j-1), r_{j+n}\}\}, \quad \forall (s,r) \in C^n \times R, \quad \forall j \in I_n, \\ &\hat{l}(s,w) = \tilde{l}(\min\{\iota^n, 3s\}, w), \quad \forall (s,w) \in C^n \times W, \\ &\hat{L}(s,w) = \tilde{L}(\max\{0, 3s - 2\iota^n\}, w), \quad \forall (s,w) \in C^n \times W, \end{aligned}$$

are again used to describe the states of the economy. Notice that $\hat{p}$ also depends on the upper and lower bounds on the set of admissible prices and $\hat{l}$ and $\hat{L}$ on the allocation of initial endowments. Define the equilibrium correspondence $\mathscr{E}: R \times W \to \prod_{i=1}^m X^i$ by $\mathscr{E}(r,w) = \{(x^1, \ldots, x^m) \in \prod_{i=1}^m X^i \mid \exists s \in C^n$ such that $(x^1, \ldots, x^m, \hat{l}(s,w), \hat{L}(s,w), \hat{p}(s,r))$ is a constrained equilibrium for the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_r, (\tilde{l}, \tilde{L}))$, $\forall (r,w) \in R \times W$.

Theorem 5.1. *Let $(\{X^i, \succeq^i\}_{i=1}^m, (\tilde{l}, \tilde{L}))$ satisfying Assumptions A1, A2, and A5 be given. Then the correspondence $\mathscr{E}: R \times W \to \prod_{i=1}^m X^i$ is upper semi-continuous.*

Proof. It will be shown that $\mathscr{E}$ is upper semi-continuous at every point $(r,w) \in R \times W$. By for example Theorem 3.4, $\mathscr{E}(r,w) \neq \emptyset$. Moreover, $\mathscr{E}$ is easily seen to be compact valued. Let $(r^t, w^t)_{t \in \mathbb{N}}$ be a sequence in $R \times W$ converging to (r,w). Let $(x^{1^t}, \ldots, x^{m^t})_{t \in \mathbb{N}}$ be a sequence with $(x^{1^t}, \ldots, x^{m^t}) \in \mathscr{E}(r^t, w^t)$. It has to be shown that this sequence has a subsequence with limit belonging to $\mathscr{E}(r,w)$. For every $t \in \mathbb{N}$ there is an $s^t \in C^n$ such that $(x^{1^t}, \ldots, x^{m^t}, \hat{l}(s^t, w^t), \hat{L}(s^t, w^t), \hat{p}(s^t, r^t))$ is a constrained equilibrium for the economy $E = (\{X^i, \succeq^i, w^{i^t}\}_{i=1}^m, P_{r^t}, (\tilde{l}, \tilde{L}))$. Consider the sequence $(x^{1^t}, \ldots, x^{m^t}, s^t)_{t \in \mathbb{N}}$. Since C^n is bounded the sequence $(s^t)_{t \in \mathbb{N}}$ is bounded. For every $h \in I_m$, since $0 \leq x^{h^t} \leq \sum_{i=1}^m w^{i^t} \to \sum_{i=1}^m w^i$, it follows also that the sequence $(x^{h^t})_{t \in \mathbb{N}}$ is bounded. Therefore the sequence $(x^{1^t}, \ldots, x^{m^t}, s^t)_{t \in \mathbb{N}}$ has a convergent subsequence with limit, say $(x^1, \ldots, x^m, s)$. It will be shown that $(x^1, \ldots, x^m, \hat{l}(s,w), \hat{L}(s,w), \hat{p}(s,r))$ satisfies all the conditions of a constrained equilibrium for the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_r, (\tilde{l}, \tilde{L}))$. Obviously $(x^1, \ldots, x^m, \hat{l}(s,w), \hat{L}(s,w), \hat{p}(s,r)) \in W \times -\mathbb{R}^{mn}_+ \times \mathbb{R}^{mn}_+ \times P_r$. Using Theorem 2.2 and the maximum theorem it follows that every δ^i is upper semi-continuous on $\mathbb{R}^n_{++} \times -\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathrm{Int}(X^i)$. Clearly, $\hat{p}, \hat{l}$, and $\hat{L}$ are continuous functions. Therefore the correspondence $\bar{\delta}^i: C^n \times R \times W$ into X^i defined by

$$\bar{\delta}^i(s,r,w) = \delta^i(\hat{p}(s,r), \hat{l}^i(s,w), \hat{L}^i(s,w), w^i), \quad \forall (s,r,w) \in C^n \times R \times W,$$

is upper semi-continuous on $C^n \times R \times W$ and hence $x^i \in \bar{\delta}^i(s,r,w)$ for every $i \in I_m$. So Condition 1 of a constrained equilibrium is satisfied. Clearly $\sum_{i=1}^m x^i = \lim_{t \to \infty} \sum_{i=1}^m w^{i^t} = \sum_{i=1}^m w^i$, thereby giving Condition 2. By Theorem 3.2 the remaining conditions of a constrained equilibrium are satisfied. Q.E.D.

A subset of a topological space is called residual if it contains a countable intersection of dense and open sets. Following the approach of Dierker [8] it is not difficult to show that the equilibrium correspondence $\mathscr{E}$ is continuous on a residual subset of $R \times W$.

Theorem 5.2. *Let $(\{X^i, \succeq^i\}_{i=1}^m, (\tilde{l}, \tilde{L}))$ satisfying Assumptions A1, A2, and A5 be given. Then the correspondence $\mathscr{E}: R \times W \to \prod_{i=1}^m X^i$ is continuous on a residual subset of $R \times W$.*

Proof. Using a theorem of Fort [10] (p. 239) and the remarks of Dierker [8] (p. 85) it is sufficient to show that $\mathscr{E}: R \times W \to \prod_{i=1}^m X^i$ is a compact valued, upper semi-continuous correspondence and that W is a separable metric space, i.e., it admits a countable dense set. These requirements are easily verified. Q.E.D.

Appendix

Theorem 2.2. *Let X^i be closed, convex, $X^i \subset \mathbb{R}^n_+$, and $Int(X^i) \neq \emptyset$, $\forall i \in I_m$. Then the correspondence $B^i: \mathbb{R}^n_{++} \times -\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times Int(X^i)$ into X^i is continuous.*

Proof. Let $(p, l^i, L^i, w^i) \in \mathbb{R}^n_{++} \times -\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathrm{Int}(X^i)$ be given. The proof of the upper semi-continuity of B^i at (p, l^i, L^i, w^i) is standard and therefore omitted. It will be shown that B^i is lower semi-continuous at (p, l^i, L^i, w^i). Let $(p^t, l^{it}, L^{it}, w^{it})_{t \in \mathbb{N}}$ be a sequence of points in $\mathbb{R}^n_{++} \times -\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathrm{Int}(X^i)$ converging to (p, l^i, L^i, w^i). Let x^i be an element of $B^i(p, l^i, L^i, w^i)$. The correspondence B^i is lower semi-continuous at the point (p, l^i, L^i, w^i) if

$(p^t, l^{it}, L^{it}, w^{it}) \to (p, l^i, L^i, w^i)$ and $x^i \in B^i(p, l^i, L^i, w^i)$ implies
there is a sequence $(x^{it})_{t \in \mathbb{N}}$ such that $x^{it} \in B^i(p^t, l^{it}, L^{it}, w^{it})$ and $x^{it} \to x^i$.

Three cases have to be considered, **1.** $p \cdot (x^i - w^i) < 0$, **2.** $p \cdot (x^i - w^i) = 0$ and $l^i = 0$, and **3.** $p \cdot (x^i - w^i) = 0$ and $l^i < 0$.

1. $p \cdot (x^i - w^i) < 0$.

First a sequence $(a^{it})_{t \in \mathbb{N}}$ in X^i converging to x^i is constructed. Then this sequence is used to construct a sequence $(c^{it})_{t \in \mathbb{N}}$ satisfying additionally that $l^{it} \leq c^{it} - w^{it} \leq L^{it}$ if t is large enough. Define the sets $J = \{j \in I_n \mid x^i_j > w^i_j\}$, $K = \{j \in I_n \mid x^i_j = w^i_j\}$, and $L = \{j \in I_n \mid x^i_j < w^i_j\}$. Since $w^i \in \mathrm{Int}(X^i)$ there exists an $\varepsilon > 0$ such that $\| y^i - w^i \|_\infty \leq \varepsilon$ implies $y^i \in X^i$. There exists a $t^1 \in \mathbb{N}$ such that for all $t > t^1$, $\| w^{it} - w^i \|_\infty \leq \varepsilon$. For $t > t^1$ define

$$a^{it}_j = w^{it}_j, \quad \forall j \in K,$$

$$a^{it}_j = \lambda^{it} x^i_j + (1 - \lambda^{it}) w^{it}_j, \quad \forall j \in J \cup L, \text{ where}$$

$$\lambda^{it} = 1, \quad \text{if } K = \emptyset, \tag{1}$$

$$\lambda^{it} = \min \left\{ \frac{\varepsilon - |w^{it}_j - w^i_j|}{\varepsilon} \,\middle|\, j \in K \right\}, \quad \text{if } K \neq \emptyset. \tag{2}$$

It will be shown that $a^{it} \in X^i$ for all $t > t^1$ and that $a^{it} \to x^i$. Clearly, for all $t > t^1$ it holds that $0 \leq \lambda^{it} \leq 1$.
If $\lambda^{it} = 1$ then by (1) and (2), $w^{it}_j = w^i_j$, $\forall j \in K$, and

$$a^{it}_j = w^{it}_j = w^i_j = x^i_j, \quad \forall j \in K,$$

$$a^{it}_j = x^i_j, \quad \forall j \in J \cup L,$$

and so $a^{it} \in X^i$.

If $\lambda^{it} < 1$ then define $b^{it} = (a^{it} - \lambda^{it}x^i)/(1 - \lambda^{it})$. It will first be shown that $b^{it} \in X^i$. Consider $j \in K$, then

$$|b_j^{it} - w_j^i| = \left| \frac{w_j^{it} - \lambda^{it}x_j^i}{1 - \lambda^{it}} - w_j^i \right| = \frac{|w_j^{it} - w_j^i|}{1 - \lambda^{it}}.$$

If $w_j^{it} = w_j^i$ then $|b_j^{it} - w_j^i| = 0 < \varepsilon$, and if $w_j^{it} \neq w_j^i$ then

$$|b_j^{it} - w_j^i| = \frac{|w_j^{it} - w_j^i|}{1 - \min\left\{ \frac{\varepsilon - |w_k^{it} - w_k^i|}{\varepsilon} \,\middle|\, k \in K \right\}} \leq \frac{|w_j^{it} - w_j^i|}{1 - \frac{\varepsilon - |w_j^{it} - w_j^i|}{\varepsilon}} = \varepsilon.$$

Consider $j \in J \cup L$, then

$$|b_j^{it} - w_j^i| = \left| \frac{\lambda^{it}x_j^i + (1 - \lambda^{it})w_j^{it} - \lambda^{it}x_j^i}{1 - \lambda^{it}} - w_j^i \right| = |w_j^{it} - w_j^i| \leq \varepsilon.$$

Hence, since $\| b^{it} - w^i \|_\infty \leq \varepsilon$, $b^{it} \in X^i$, and since $(1 - \lambda^{it})b^{it} + \lambda^{it}x^i = a^{it}$ it holds by the convexity of X^i that $a^{it} \in X^i$.

Moreover, $\lambda^{it} = \min\left\{ \frac{\varepsilon - |w_j^{it} - w_j^i|}{\varepsilon} \,\middle|\, j \in K \right\} \to 1$ and so

$$\begin{aligned} a_j^{it} &= w_j^{it} && \to w_j^i = x_j^i, && \forall j \in K, \\ a_j^{it} &= \lambda^{it}x_j^i + (1 - \lambda^{it})w_j^{it} && \to x_j^i, && \forall j \in J \cup L. \end{aligned}$$

So there exists a $t^2 \in \mathbb{N}$ such that $t^2 \geq t^1$ and for all $t > t^2$

$$a_j^{it} > w_j^{it}, \quad \forall j \in J, \quad \text{and } a_j^{it} < w_j^{it}, \quad \forall j \in L.$$

Take $t > t^2$. For every $j \in J$ let $\mu_j^{it} = L_j^{it}/(a_j^{it} - w_j^{it})$, then $\mu_j^{it} \geq 0$ since $a_j^{it} - w_j^{it} > 0$ and $L_j^{it} \geq 0$. For every $j \in L$ let $\mu_j^{it} = l_j^{it}/(a_j^{it} - w_j^{it})$, then $\mu_j^{it} \geq 0$ since $a_j^{it} - w_j^{it} < 0$ and $l_j^{it} \leq 0$. Finally, let $\mu_*^{it} = \min(\{\mu_j^{it} | j \in J \cup L\} \cup \{1\})$. Clearly, $0 \leq \mu_*^{it} \leq 1$. Next, let $c^{it} = w^{it} + \mu_*^{it}(a^{it} - w^{it})$. Since $a^{it}, w^{it} \in X^i$ and by the convexity of X^i it holds that $c^{it} \in X^i$. Moreover,

$$\begin{aligned} & c_j^{it} - w_j^{it} = \mu_*^{it}(a_j^{it} - w_j^{it}) \leq \mu_j^{it}(a_j^{it} - w_j^{it}) = L_j^{it}, && \forall j \in J, \\ & c_j^{it} - w_j^{it} = \mu_*^{it}(a_j^{it} - w_j^{it}) \geq 0 \geq l_j^{it}, && \forall j \in J, \\ & c_j^{it} - w_j^{it} = \mu_*^{it}(a_j^{it} - w_j^{it}) = 0 \text{ and so } l_j^{it} \leq c_j^{it} - w_j^{it} \leq L_j^{it}, && \forall j \in K, \\ & c_j^{it} - w_j^{it} = \mu_*^{it}(a_j^{it} - w_j^{it}) \geq \mu_j^{it}(a_j^{it} - w_j^{it}) = l_j^{it}, && \forall j \in L, \\ & c_j^{it} - w_j^{it} = \mu_*^{it}(a_j^{it} - w_j^{it}) \leq 0 \leq L_j^{it}, && \forall j \in L. \end{aligned}$$

Further,

$$\mu_j^{it} = \frac{L_j^{it}}{\alpha_j^{it} - w_j^{it}} \to \frac{L_j^i}{x_j^i - w_j^i} \geq \frac{x_j^i - w_j^i}{x_j^i - w_j^i} = 1, \quad \forall j \in J,$$

$$\mu_j^{it} = \frac{l_j^{it}}{a_j^{it} - w_j^{it}} \to \frac{l_j^i}{x_j^i - w_j^i} \geq \frac{x_j^i - w_j^i}{x_j^i - w_j^i} = 1, \quad \forall j \in L.$$

So $\mu_*^{it} \to 1$ and therefore $c^{it} = w^{it} + \mu_*^{it}(a^{it} - w^{it}) \to w^i + x^i - w^i = x^i$. Moreover, $p^t \cdot (c^{it} - w^{it}) \to p \cdot (x^i - w^i) < 0$. Therefore, there exists a $t^3 \in \mathbb{N}$ such that $t^3 \geq t^2$ and $t > t^3$ implies $p^t \cdot (c^{it} - w^{it}) < 0$. Hence, for $t > t^3$, $c^{it} \in B^i(p^t, l^{it}, L^{it}, w^{it})$. Construct the sequence $(x^{it})_{t \in \mathbb{N}}$ as follows:

$$\begin{aligned} & t \leq t^3, \quad x^{it} \text{ is an arbitrary element of } B^i(p^t, l^{it}, L^{it}, w^{it}); \\ & t > t^3, \quad x^{it} = c^{it}. \end{aligned}$$

This sequence clearly has all the desired properties.

2. $p\cdot(x^i - w^i) = 0$ and $l^i = 0$.
Since $x^i - w^i \geq l^i$ and $l^i = 0$, $x^i \geq w^i$. Using $p \gg 0$ this implies $x^i = w^i$. Obviously $w^{it} \in B^i(p^t, l^{it}, L^{it}, w^{it})$. Moreover, $w^{it} \rightarrow w^i = x^i$. Define the sequence $(x^{it})_{t\in\mathbb{N}}$ by $x^{it} = w^{it}$, $\forall t \in \mathbb{N}$. This sequence clearly has all the desired properties.

3. $p\cdot(x^i - w^i) = 0$ and $l^i < 0$.
Define the non-empty set $J = \{j \in I_n | l^i_j < 0\}$. Since $w^i \in \mathrm{Int}(X^i)$ there exists an $\alpha > 0$ such that $\| y^i - w^i \|_\infty \leq \alpha$ implies $y^i \in X^i$. There exists a $t^1 \in \mathbb{N}$ such that for all $t > t^1$, $\| w^{it} - w^i \|_\infty \leq \frac{1}{2}\alpha$ and $l^{it}_j < 0$, $\forall j \in J$. Take $t > t^1$. Define $\varepsilon^t > 0$ by $\varepsilon^t = \min(\{-l^{it}_j | j \in J\} \cup \{\frac{1}{2}\alpha\})$ and define $\underline{w}^{it}$ by $\underline{w}^{it}_j = w^{it}_j - \varepsilon^t$, $\forall j \in J$, and $\underline{w}^{it}_j = w^{it}_j$, $\forall j \in I_n \backslash J$. Then $\underline{w}^{it}$ has the following properties,

$$\underline{w}^{it} \in X^i, \quad l^{it} \leq \underline{w}^{it} - w^{it} \leq 0 \leq L^{it}, \quad \text{and } p^t\cdot\underline{w}^{it} < p^t\cdot w^{it}. \tag{3}$$

Moreover, $\underline{w}^{it} \rightarrow \underline{w}^i$ where, for ε equal to $\min(\{-l^i_j | j \in J\} \cup \{\frac{1}{2}\alpha\})$, $\underline{w}^i_j = w^i_j - \varepsilon$, $\forall j \in J$, and $\underline{w}^i_j = w^i_j$, $\forall j \in I_n \backslash J$, so

$$\underline{w}^i \in X^i, \quad l^i \leq \underline{w}^i - w^i \leq 0 \leq L^i, \quad \text{and } p\cdot\underline{w}^i < p\cdot w^i.$$

Consider the sequence $(c^{it})_{t\in\mathbb{N}}$ defined as in Part 1 of the proof. It may be assumed that the elements of this sequence satisfy

$$c^{it} \in X^i, \quad l^{it} \leq c^{it} - w^{it} \leq L^{it}, \quad \text{and } c^{it} \rightarrow x^i. \tag{4}$$

If $p^t\cdot c^{it} > p^t\cdot w^{it}$ then define λ^{it} by

$$\lambda^{it} = \frac{p^t\cdot w^{it} - p^t\cdot\underline{w}^{it}}{p^t\cdot c^{it} - p^t\cdot\underline{w}^{it}} \tag{5}$$

and if $p^t\cdot c^{it} \leq p^t\cdot w^{it}$ then define $\lambda^{it} = 1$. Clearly $0 < \lambda^{it} \leq 1$. Define $d^{it} = \underline{w}^{it} + \lambda^{it}(c^{it} - \underline{w}^{it})$. Using the convexity of X^i, $d^{it} \in X^i$. By (3) and (4)

$$d^{it} - w^{it} = \lambda^{it}(c^{it} - w^{it}) + (1 - \lambda^{it})(\underline{w}^{it} - w^{it}) \geq l^{it},$$
$$d^{it} - w^{it} = \lambda^{it}(c^{it} - w^{it}) + (1 - \lambda^{it})(\underline{w}^{it} - w^{it}) \leq L^{it}.$$

If $p^t\cdot c^{it} > p^t\cdot w^{it}$ then by (5)

$$p^t\cdot d^{it} = \left(\frac{p^t\cdot w^{it} - p^t\cdot\underline{w}^{it}}{p^t\cdot c^{it} - p^t\cdot\underline{w}^{it}}\right)p^t\cdot c^{it} + \left(\frac{p^t\cdot c^{it} - p^t\cdot w^{it}}{p^t\cdot c^{it} - p^t\cdot\underline{w}^{it}}\right)p^t\cdot\underline{w}^{it} = p^t\cdot w^{it},$$

and if $p^t\cdot c^{it} \leq p^t\cdot w^{it}$ then because $\lambda^{it} = 1$

$$p^t\cdot d^{it} = p^t\cdot c^{it} \leq p^t\cdot w^{it}.$$

So $d^{it} \in B^i(p^t, l^{it}, L^{it}, w^{it})$. Using $c^{it} \rightarrow x^i$ and $p\cdot x^i = p\cdot w^i$,

$$\frac{p^t\cdot w^{it} - p^t\cdot\underline{w}^{it}}{p^t\cdot c^{it} - p^t\cdot\underline{w}^{it}} \rightarrow \frac{p\cdot w^i - p\cdot\underline{w}^i}{p\cdot w^i - p\cdot\underline{w}^i} = 1$$

and so $\lambda^{it} \rightarrow 1$. Consequently, $d^{it} \rightarrow \underline{w}^i + (x^i - \underline{w}^i) = x^i$. Construct the sequence $(x^{it})_{t\in\mathbb{N}}$ as follows:

$$t \leq t^1, \quad x^{it} \text{ is an arbitrary element of } B^i(p^t, l^{it}, L^{it}, w^{it});$$
$$t > t^1, \quad x^{it} = d^{it}.$$

This sequence clearly has all the desired properties. Q.E.D.

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