

Nash implementation in Stochastic Social Choice

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Abstract

This paper studies the implementation problem when randomization over alternatives is allowed. We show that, when there are at least three agents, Maskin monotonicity is necessary and sufficient for any (possibly non-ordinal) social choice correspondences to be Nash implementable. Since Maskin monotonicity is an easy-to-check condition, this result contrasts with various complex conditions characterizing Nash implementability in deterministic social choice and in general environments.

Keywords: Nash Implementation, Maskin monotonicity, Virtual Implementation, No-veto power, Stochastic Social Choice.

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1 Introduction

Randomization over alternatives has received a considerable amount of attention in the theory of social choice. The stochastic approach has been motivated by the desire to escape the impossibility results obtained in the deterministic framework. Indeed, various interesting social choice correspondences (SCCs) that select lotteries were provided (e.g., Gibbard, 1977; Barbera, 1978, 1979; Pattanaik and Peleg, 1986). From the viewpoint of implementation theory, a great deal of work has been devoted on analyzing virtual Nash implementation (e.g., Matsushima, 1988; Abreu and Sen, 1991). On the other hand, only a few efforts were carried out in studying (exact) Nash implementation in stochastic social choice. However, all mechanisms in the virtual implementation literature have a drawback that they always put positive probabilities to all alternatives, so any inefficient alternative can be realized as a social outcome.¹ Therefore, though the virtual implementation approach has succeeded in obtaining some remarkable positive results, it is also important to consider the original Nash implementation notion.²

As in the deterministic framework, *Maskin monotonicity* has played a key role on Nash implementation in the stochastic framework. Maskin's (1999) theorem implies that *Maskin monotonicity* is necessary for Nash implementation. On sufficiency, we can say the following by summarizing Maskin's theorem and the results on stochastic social choice by Matsushima (1988), Bochet (2005), and Benoit and Ok (2004): a social choice correspondence (SCC) is Nash implementable if it is *Maskin monotonic* and satisfies any one of the following additional conditions:

- **No veto-power:** if a lottery is the best for at least $n - 1$ agents among n agents, then it is chosen;
- **Totally-mixedness:** all alternatives always have positive probabilities;
- **Non-mixedness:** one and only one alternative receives probability one.

¹Bochet and Maniquet (2005) recently obtained a necessary and sufficient condition for virtual Nash implementable with an "admissible" support. Their condition jointly restricts a social choice function and an admissible support function in a monotonicity fashion.

²For other comments on virtual implementation, see Jackson (2001).

However, most of interesting *Maskin monotonic* SCCs in this literature, such as the “random dictatorship” or the “status-quo lower bound SCC”, satisfies none of these additional conditions. Therefore, these results do not help us check Nash implementability of such SCCs. This motivates us to obtain a more general condition for Nash implementation.³ We show that, for societies with at least three agents, *Maskin monotonicity* is necessary and sufficient for Nash implementation. Thus, there is no gap between necessity and sufficiency. Besides the generality of this theorem as a full characterization, it is general in three ways: it applies not only to social choice *functions*, but also to social choice *correspondences*; ⁴ it does not rely on any ordinality condition; ⁵ and finally it applies to arbitrary preference domain. Since *Maskin monotonicity* is an easy-to-check condition, this theorem provides us with a quite convenient way to check Nash implementability of SCCs. This contrasts with various complex conditions characterizing Nash implementability in deterministic social choice (e.g., Danilov, 1992; Yamato, 1992) and in general environments (e.g., Moore and Repullo, 1990).

This paper is organized as follows: Section 2 introduces the model; Section 3 explains Nash implementation and reviews some results in the literature; Section 4 provides our theorem; Section 5 concludes the discussion.

2 The model

Let $N \equiv \{1, 2, \dots, n\}$ be a finite set of agents with $n \geq 3$. There is a finite set of pure alternatives $A \equiv \{a_1, a_2, \dots, a_\ell\}$. Let $\mathcal{L} \equiv \{x \in \mathbb{R}_+^A : \sum_{a \in A} x_a = 1\}$ be the set of lotteries over A . In lottery $x \equiv (x_a)_{a \in A} \in \mathcal{L}$, alternative a occurs with probability x_a . Abusing notation, we write a both for the pure

³Moore and Repullo (1990) provide a necessary and sufficient condition for Nash implementation in general environments. However, checking their condition is hard.

⁴As far as the authors know, this study is the first one that analyzes SCCs in stochastic social choice. Examples of interesting SCCs are the status-quo lower bound correspondence and the Pareto correspondence.

⁵Ordinality states the invariance of outcomes between two preference profiles that agree on the rankings over pure alternatives. This condition is usually imposed in stochastic social choice (e.g., Gibbard, 1977; Matsushima, 1988; Abreu and Sen, 1991; Bochet, 2004; Benoit and Ok, 2004). Ordinality is a demanding condition. For example, when there are two agents and three pure alternatives, any ordinal SCC has at most 36 ($3! \times 3!$) images, though there are infinitely many types of preferences for each agent. Abreu and Sen (1991) study virtual implementation of non-ordinal SCCs.

alternative $a \in A$ and for the lottery $x \in \mathcal{L}$ with $x_a = 1$. Let $\text{supp}x \equiv \{a \in A \mid x_a > 0\}$ be the support of a lottery $x \in \mathcal{L}$.

The set of admissible preference profiles over \mathcal{L} is \mathcal{R} . For each $R \equiv (R_1, R_2, \dots, R_n) \in \mathcal{R}$ and each $i \in N$, R_i denotes i 's preference over \mathcal{L} . The strict and indifference relations associated with R_i are P_i and I_i , respectively. We assume that each $R \in \mathcal{R}$ satisfies the following standard conditions for all $i \in N$:

Continuity: For each $x \in X$, both sets $\{y \in X : y P_i x\}$ and $\{y \in X : x P_i y\}$ are open.

Strictness: no two alternatives are indifferent. Formally, for each $(a, b) \in A^2$, if $a R_i b$, then $a P_i b$ or $a = b$.

Monotonicity in probability: Shifts in probability to strictly preferred alternatives yield strictly preferred lotteries. Formally, for each $k \in \{1, \dots, \ell\}$, let $a(R_i, k) \in A$ be i 's k -th preferred alternative at R , i.e. $|\{b \in A : b R_i a(R_i, k)\}| = k$. Note that *strictness* ensures the uniqueness of each $a(R_i, k)$. Then, if two lotteries $x, y \in \mathcal{L}$ are such that for each $k^* \in \{1, \dots, \ell\}$,

$$\sum_{k \leq k^*} x_{a(R_i, k)} \geq \sum_{k \leq k^*} y_{a(R_i, k)},$$

then for each $R' \in \mathcal{R}(R)$, we have that $x R'_i y$; and whenever the inequality is strict for some k^* , then $x P'_i y$.

Given $R \in \mathcal{R}$, $i \in N$, and $x \in \mathcal{L}$, let $LC(R_i, x)$ be the lower contour set of $R_i \in N$ at x , i.e.,

$$LC(R_i, x) \equiv \{y \in \mathcal{L} : x R_i y\}.$$

Also, let $\text{top}(R_i) \in A$ be the pure alternative such that for each $x \in \mathcal{L}$, $\text{top}(R_i) R_i x$. Given $R \in \mathcal{R}$ and $x \in \mathcal{L}$, a preference profile $R' \in \mathcal{R}$ is a *Maskin monotonic transformation of R at x* if for each $i \in N$, $LC(R_i, x) \subseteq LC(R'_i, x)$. Let $MT(R, x)$ be the set of Maskin monotonic transformations of R at x .

A *social choice correspondence* (SCC) is a non-empty valued correspondence f from \mathcal{R} to \mathcal{L} that associates with each preference profile R a non-empty set of lotteries $f(R)$. A SCC is a *social choice function* (SCF) if it is single-valued.

Remark 1 *Our model can be interpreted as a public good model. Then, A is the set of types of public goods, each $x \in \mathcal{L}$ is a budget distribution over A , and each $i \in N$ has a preference over budget distributions.⁶ This interpretation is taken by Schummer (1999).*

3 Nash implementation

A *game form* is a pair $G \equiv (M, g)$, where $M \equiv \times_{i \in N} M_i$, M_i is the message space of $i \in N$, and $g : M \rightarrow \mathcal{L}$ is an outcome function that associates with each message profile a lottery. A typical message profile is $m \equiv (m_1, \dots, m_n) \in M$. A *game* is a pair of a game form and a preference profile. Given a game form $G \equiv (M, g)$ and $R \in \mathcal{R}$, $m \in M$ is a *Nash equilibrium* of the game (G, R) if for each $i \in N$ and each $m'_i \in M_i$,

$$g(m) R_i g(m'_i, m_{-i}).$$

Let $NE(G, R)$ be the set of Nash equilibria of the game (G, R) and let

$$g(NE(G, R)) \equiv \{x \in \mathcal{L} : \exists m \in NE(G, R), x = g(m)\}$$

be the set of Nash equilibrium outcomes in (G, R) .

Nash implementability: There exists a game form G such that for each $R \in \mathcal{R}$, $f(R) = g(NE(G, R))$.

The following condition states the preservation of a chosen outcome with respect to the improvement of its relative position (Maskin, 1999):

Maskin monotonicity: For each $R \in \mathcal{R}$, each $x \in f(R)$ and each $R' \in MT(R, x)$, we have $x \in f(R')$.

The next condition states that if a lottery is the best for at least $n - 1$ number of agents, then it should be chosen:

No-veto power: For each $R \in \mathcal{R}$ and each $x \in L$,

$$|\{i \in N : x = \text{top}(R_i)\}| \geq n - 1 \text{ implies that } x \in f(R).$$

⁶Note that *monotonicity in probabilities* requires that each agent has a “best” public good, and he/she wants all the budget to be allocated to the good. An example of such situations is that A is the set of locations of a public service (e.g., library, police) and each agent wants a service at a closer location to receive a higher budget.

For the sake of completeness, we recall here—proposition 1 and 2—Maskin’s (1999) classical results on necessity and sufficiency of Nash implementation:

Proposition 1: *If a SCC is Nash implementable, then it is Maskin monotonic.*

Proposition 2: *If a SCC satisfies Maskin monotonicity and no veto-power, then it is Nash implementable.*

There are alternative conditions to *no veto-power* for Proposition 2. A SCC is *totally-mixed* if for each $R \in \mathcal{R}$ and each $a \in A$, $f_a(R) > 0$. A SCC is *non-mixed* if for each $R \in \mathcal{R}$, there exists $a \in A$ such that $f_a(R) = 1$. Note that any *non-mixed* SCC is in fact a SCF.

Proposition 3: *If a SCC is Maskin monotonic and is totally-mixed, then it is Nash implementable.*

Proof: The proof is based on Matsushima (1988, Section 5.1), who points out that, when preferences are linear, any *totally-mixed* SCF is *Nash implementable*.⁷

Let f be a *Maskin monotonic* and *totally-mixed* SCC. Let $f': \mathcal{R} \rightarrow \text{int}\mathcal{L}$ be such that for each $R \in \mathcal{R}$, $f'(R) = f(R)$. Since no one has a best lottery in $\text{int}\mathcal{L}$, f' vacuously satisfies no-veto power.⁸ By Maskin’s theorem, there exists a game form that Nash implements f' . Obviously this game form also Nash implements f .

We remark that when R only contains linear preferences, f' vacuously satisfies *Maskin monotonicity*. In this case, *Maskin monotonicity* of f is not needed.⁹

Q.E.D.

⁷Matsushima also imposes *no-veto power*. However, as our proof shows, it is redundant.

⁸We are slightly abusing language here. Formally speaking, we defined no-veto power (and Maskin monotonicity) for SCCs whose range is \mathcal{L} . Hence these conditions need to be redefined for SCCs whose range is $\text{int}\mathcal{L}$.

⁹See Matsushima, 1988, Section 5.1. Any SCC f that maps to the interior of the probability simplex is trivially *Maskin monotonic* when preferences over lotteries are linear. For more general preferences satisfying the *monotonicity in probability* axiom, this statement is true if f is ordinal.

The following proposition is a version of the results in Bochet (2004, Theorem 2) and Benoit and Ok (2004, Theorem 1):¹⁰

Proposition 4: *If a SCC is Maskin monotonic and is non-mixed, then it is Nash implementable.*

Summarizing Proposition 2, 3 and 4, we can say that if a *Maskin monotonic* SCC satisfies any one of *no veto-power*, *totally-mixedness*, and *non-mixedness*, then it is *Nash implementable*. However, the three conditions are not widely met. Following are such examples:

w-Random dictatorship, f^w (Gibbard, 1977): There exists a “weight vector” on agents $w \in \left\{ w \in \mathbb{R}_+^N : \sum_{i \in N} w_i = 1 \right\}$ such that for each $R \in \mathcal{R}$,

$$f^w(R) \equiv \left\{ x \in \mathcal{L} : x = \sum_{i \in N} (w_i \cdot \text{top}(R_i)) \right\}.$$

Obviously, f^w violates *no-veto power* if there are $R \in \mathcal{R}$, $a \in A$ and $j \in N$ such that $\text{top}(R_j) \neq a$ and for each $i \neq j$, $\text{top}(R_i) = a$. Also, it is neither *totally-mixed* nor *non-mixed* if $w > 0$ and there is $R \in \mathcal{R}$ such that $2 \leq |\{a \in A : \text{for some } i \in N, a = \text{top}(R_i)\}| < \ell$. We remark that f^w is *Maskin monotonic* in many cases (e.g., \mathcal{R} contains only linear preferences).

Status-quo lower bound correspondence, f^s : There exists a “status-quo” alternative $a^* \in A$ such that for each $R \in \mathcal{R}$,

$$f^s(R) \equiv \{x \in \mathcal{L} : \text{for each } i \in N, x R_i a^*\}.$$

This SCC is not *totally-mixed* in general since for each $R \in \mathcal{R}$, $a^* \in f^s(R)$. It violates *no-veto power* if there are $R \in \mathcal{R}$, $a \in A$ and $j \in N$ such that $a^* P_j a$ and for each $i \neq j$, $a = \text{top}(R_i)$. Also, it violates *non-mixedness* if there are $R \in \mathcal{R}$ and $x \in f^s(R)$ such that $2 \leq |\text{supp}x|$. On the other hand, f^s is in general *Maskin monotonic*.

Pareto correspondence, f^p : For each $R \in \mathcal{R}$, let

$$f^p \equiv \{x \in \mathcal{L} : \nexists y \in \mathcal{L} \text{ such that } \forall i \in N, y R_i x \text{ and } \exists j \in N \text{ such that } y P_j x\}.$$

¹⁰Bochet (2004) and Benoit and Ok (2004) use a somewhat different framework than ours. However, their proofs in fact imply this proposition.

This SCC is *Maskin monotonic* when preferences are convex. It satisfies *no-veto power* but violates the other two conditions.

The intersection of f^s and f^p : This SCC is well-defined when preferences are convex. Then, it is *Maskin monotonic*. However, it violates the other conditions unless the preference domain is so small that the conditions are vacuously satisfied.

4 Equivalence theorem

Our theorem shows that *Maskin monotonicity* is equivalent to *Nash implementability*. Thus, the additional conditions in Propositions 2, 3 and 4 are redundant. Since *Maskin monotonicity* is an easy-to-check condition, this theorem provides a convenient way to check *Nash implementability* of any SCC.

Theorem: *A SCC is Nash implementable if and only if it is Maskin monotonic.*

Proof: the only if part follows from proposition 1. Let us prove the opposite direction. Consider an arbitrary *Maskin monotonic* SCC f . We construct the following game form. For each $i \in N$, the message space is,

$$M_i \equiv \mathcal{R} \times \mathcal{L} \times \{(a, b) \in A^2 : a \neq b\} \times \mathbb{N},$$

where a typical message is denoted $m_i \equiv (R, x, a, b, n) \in M_i$. Let $M \equiv \times_{i \in N} M_i$. Next, we define the outcome function. Consider an arbitrary message profile $m \in M$.

Rule 1: Consider the case that there exists (R, x, a, b, n) with $x \in f(R)$, and for each $i \in N$, $m_i = (R, x, a, b, n)$. Then, let $g(m) = x$.

Rule 2: Consider the case that there exists (R, x, a, b, n) with $x \in f(R)$, $m_i = (R, x, a, b, n)$ for each $i \neq j$ and $m_j \neq m_i$. We write $m_j = (R', x', a', b', n')$. Let q be the infinite dimensional vector defined by,

$$q \equiv \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \dots \right).$$

Rule 2-1: If $x' \in LC(R_j, x)$, $a' \in \text{supp}x'$ and $a' P_j b'$, then

$$g(m) = x' + q_{n'} \cdot x'_{a'} \cdot (b' - a').$$

Rule 2-2: If x' is the worst lottery for R_j (so it is a pure alternative), $x' \neq a'$ and $x P_j (q_{n'}x' + (1 - q_{n'})a')$, then

$$g(m) = q_{n'}x' + (1 - q_{n'})a'.$$

Note that the cases for Rule 2-1 and 2-2 are exclusive. Indeed, if x' is such that there are $a' \in \text{supp}x'$ and $b' \in A$ for which $a' P_j b'$, then x' is not worst for R_j . Also, note that in both cases, $g(m)$ is not a pure alternative.

Rule 2-3: Otherwise, $g(m) = x$.

Rule 3: In all other cases, let

$$g(m) = \left(1 - \frac{1}{1 + n^{i^*}}\right) a^{i^*} + \frac{1}{1 + n^{i^*}} b^{i^*},$$

where $i^* \equiv \{i \in N: n^i \geq n^j \text{ for all } j \in N\}$.

We show that this game form implements f . Let $R \in \mathcal{R}$ be the true preference profile.

Step 1: $f(R) \subseteq g(NE(G, R))$

Let $x \in f(R)$ and let $m \in M$ be such that for each $i \in N$,

$$m_i \equiv (R, x, a, b, n) \text{ where } a, b \text{ and } n \text{ are arbitrary.}$$

It suffices to show that $m \in NE(G, R)$. Consider arbitrary j 's deviation from m using $m'_j = (R', x', a', b', n')$ such that $g(m'_j, m_{-j}) \neq g(m)$.

When Rule 2-1 applies, we have

$$g(m'_j, m_{-j}) = x' + q_{n'} \cdot x'_{a'} \cdot (b' - a') \text{ and } a' P_j b'.$$

Thus, by *monotonicity in probability*, $g(m) = x R_j x' P_j g(m'_j, m_{-j})$. Hence, this deviation is not profitable.

When Rule 2-2 applies, we obtain that,

$$x P_j (q_{n'}x' + (1 - q_{n'})a') \text{ and } g(m'_j, m_{-j}) = (q_{n'}x' + (1 - q_{n'})a')$$

Thus,

$$g(m) = x P_j (q_{n'} x' + (1 - q_{n'}) a') = g(m'_j, m_{-j}).$$

Hence, this deviation is not profitable for agent j . Therefore, $m \in NE(G, R)$.

Step 2: $g(NE(G, R)) \subseteq f(R)$

Let $m \in NE(G, R)$. We need to show that $g(m) \in f(R)$. There are three cases to consider.

Case 1: $g(m)$ is given by Rule 3.

Let n^{i*} be the number that i^* reported at m_{i^*} . By monotonicity in probabilities, for each $j \in N$, there is $n' > n^{i*}$ such that for $m'_j = (R', x', a(R_j, 1), a(R_j, 2), n')$ (R' and x' arbitrary), we obtain that $g(m'_j, m_{-j}) P_j g(m)$. Hence any agent wants to deviate from m . Therefore, there is no equilibrium coming from Rule 3.

Case 2: $g(m)$ is given by Rule 2.

Let j be the agent whose message is different from others'. We write $m_j = (R^j, x^j, a^j, b^j, n^j)$ and $m_i = (\bar{R}, \bar{x}, \bar{a}, \bar{b}, \bar{n})$ for $i \neq j$. In this case, $\bar{x} \in f(\bar{R})$.

The proof proceeds in two claims. First, we prove that $g(m) = \bar{x}$. Next, we show that $R \in MT(\bar{R}, \bar{x})$. Since $\bar{x} \in f(\bar{R})$, by *Maskin monotonicity*, these two claims establish that $g(m) \in f(R)$.

Claim 1: $g(m) = \bar{x}$

If $g(m)$ is not the best lottery for some R_i with $i \neq j$, then by *monotonicity in probability*, there exists n' such that i could gain by reporting $m'_i = (R', x', a(R_i, 1), a(R_i, 2), n')$ (with R' and x' arbitrary) and trigger Rule 3. Hence for each $i \neq j$, $g(m)$ is the best lottery for R_i . Thus, $g(m)$ is a pure alternative. When Rule 2-1 or 2-2 applies, $g(m)$ cannot be a pure alternative. Hence, $g(m)$ was in fact given by Rule 2-3. Thus, $g(m) = \bar{x}$.

Claim 2: $R \in MT(\bar{R}, \bar{x})$

For each $i \neq j$, since \bar{x} is the best lottery for R_i , we have that $LC(\bar{R}_i, \bar{x}) \subseteq \mathcal{L} = LC(R_i, \bar{x})$. It remains to show that $LC(\bar{R}_j, \bar{x}) \subseteq LC(R_j, \bar{x})$. Suppose by contradiction that there is $x' \in \mathcal{L}$ such that $\bar{x} \bar{R}_j x'$ and $x' P_j \bar{x}$. We consider two cases: (i) x' is the worst lottery for \bar{R}_j and (ii) Otherwise.

Consider first case (i). By $x' P_j \bar{x}$, we have that $x' \neq \bar{x}$. Thus, by uniqueness of the worst lottery, $\bar{x} \bar{P}_j x'$. Since $\bar{x} \bar{P}_j x'$ and $x' P_j \bar{x}$, there exist $n' \neq \bar{n}$ and $a' \neq x'$ such that,

$$\bar{x} \bar{P}_j (q_{n'} x' + (1 - q_{n'}) a') \text{ and } (q_{n'} x' + (1 - q_{n'}) a') P_j \bar{x}.$$

Thus, agent j could gain by reporting any (R', x', a', b', n') (with R' and x' arbitrary) and trigger Rule 2-2. Now, consider case (ii). Since x' is not the worst lottery for \bar{R}_j , there exists $a' \in \text{supp} x'$ and $b' \in A$ such that $a' \bar{P}_j b'$. Since $x' P_j \bar{x}$, there exists $n' \neq \bar{n}$ such that,

$$(x' + q_{n'} \cdot x_{a'} \cdot (b' - a')) P_j \bar{x}.$$

Thus, j could gain by reporting the message (R', x', a', b', n') (with R' arbitrary) and trigger Rule 2-1. In either case, we derived a contradiction, so $LC(\bar{R}_j, \bar{x}) \subseteq LC(R_j, \bar{x})$. Therefore, $R \in MT(\bar{R}, \bar{x})$.

Case 3: $g(m)$ is given by Rule 1

We write $m_i = (\bar{R}, \bar{x}, \bar{a}, \bar{b}, \bar{n})$ for all $i \in N$. Hence, $g(m) = \bar{x} \in f(\bar{R})$. By a way identical to what was shown in claim 2 case (ii), we can show that for each $i \in N$, $LC(\bar{R}_i, \bar{x}) \subseteq LC(R_i, \bar{x})$. That is, $R \in MT(\bar{R}, \bar{x})$ and by Maskin monotonicity, it follows that $\bar{x} \in f(R)$.

Q.E.D.

5 Conclusion

We showed that *Maskin monotonicity* is necessary and sufficient for Nash implementation in stochastic social choice. Since *Maskin monotonicity* is an easy-to-check condition, this contrasts with complex (and in fact stronger) conditions characterizing *Nash implementability* in deterministic social choice (e.g., Danilov, 1992; Yamato, 1992) and in general environments (e.g., Moore and Repullo, 1990).¹¹

The idea of the mechanism we constructed is based on Maskin's (1999) mechanism. Hence, it is not immune to the usual criticisms addressed to

¹¹Danilov and Yamato's conditions are stronger versions of *Maskin monotonicity*. Moore and Repullo's condition consists of some subconditions including *Maskin monotonicity*. Our theorem implies that other subconditions are implied by *Maskin monotonicity* in stochastic social choice.

canonical mechanisms based on Maskin's mechanism.¹² Also, our mechanism depends on an artificial device that no agent can find a best response given others' messages in Rule 3.¹³ Improving these parts of our mechanism and finding natural mechanisms for each *Maskin monotonic* SCC are interesting future works. We close the discussion by providing a simple mechanism that Nash implements the status-quo lower bound SCC:¹⁴ For each $i \in N$, let $M_i \equiv \mathcal{R} \times \mathcal{L}$. Let $g : M \rightarrow \mathcal{L}$ be such that for each $m \in M$,

$$\begin{aligned} g(m) &= x \quad \text{if } \exists (R, x) \in \mathcal{R} \times \mathcal{L}, \forall i \in N, m_i = (R, x) \text{ and } x \in f^s(R), \\ &= a^* \quad \text{otherwise.} \end{aligned}$$

We omit the easy proof that this mechanism Nash implements f^s .

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¹²See, Jackson (2001) for such critics.

¹³This requirement on mechanisms is called the "best response property" (Jackson, Palfrey, and Srivastava, 1994). However, many mechanisms in stochastic social choice violate this property (e.g., Abreu and Sen, 1991; Benoit and Ok, 2004; Bochet, 2004)

¹⁴This mechanism is a transcription of the mechanism implementing the "individually rational SCC" in the matching problem by Kara and Sonmez (1996).

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