

Secure implementation in allotment economies ^{*}

Olivier Bochet [†]and Toyotaka Sakai [‡]

August 3, 2007

Abstract

An allocation rule is *securely implementable* if it is strategy-proof and has no “bad” Nash equilibrium in its associated direct revelation game (Saijo, Sjöström, and Yamato, 2007). We study this implementability notion in allotment economies with single-peaked preferences (Sprumont, 1991). We show that the equal division rule is the unique symmetric and securely implementable rule. The uniform rule, which is central in the literature, is not securely implementable, though it is symmetric and strategy-proof. We next investigate the structure of the set of Nash equilibria of the direct revelation game associated with the uniform rule. We show that any “bad” Nash equilibrium is blocked by a credible coalitional deviation that realizes the true uniform allocation, while any “good” Nash equilibrium is robust against any coalitional deviation. Thus the impossibility of securely implementing the uniform rule can be resolved by allowing pre-play communication among players.

Keywords: Secure implementation, Strategy-proofness, Uniform rule, Nash implementation, Coalition-proof Nash equilibrium, Single-peaked preference, Fair allocation.

JEL codes: C72, D63, D61, C78, D71.

^{*}The authors thank Ryusuke Shinohara and Takuma Wakayama for helpful advices. Part of this paper was written when O. Bochet was visiting Yokohama National University. The visit was supported by the Japan Society for the Promotion of Science under Grant-in-Aid for Young Scientists (B) 18730132, which is gratefully acknowledged.

[†]Maastricht University and CORE; Department of Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht; o.bochet@algec.unimaas.be; <http://www.personeel.unimaas.nl/o.bochet>

[‡]Correspondence: Faculty of Economics and IGSSS, Yokohama National University, Hodogaya, Yokohama 240-8501, Japan; toyotaka@ynu.ac.jp; http://www.geocities.jp/toyotaka_sakai/

1 Introduction

Strategy-proofness is a central condition in implementation and mechanism theory. An allocation rule is strategy-proof if truth-telling is a dominant strategy for everyone in its associated direct revelation game. However, even if a rule is strategy-proof, agents may play some “bad” Nash equilibrium so that undesirable outcomes are realized.¹ To avoid this problem, a seminal work by Saijo, Sjöström, and Yamato (2007) (hereafter, SSY) suggests to impose the additional requirement for a strategy-proof rule to allow no such bad Nash equilibrium. They call this property *secure implementability*. More formally saying, a rule is securely implementable if it is strategy-proof and its associated direct revelation game has no Nash equilibrium that realizes an outcome that differs from the outcome obtained under truthful revelation. Our purpose is to examine the possibility of secure implementation in the problem of fairly allocating a divisible resource to agents with single-peaked preferences (Sprumont, 1991).

The central allocation rule in this context is the uniform rule (Benassy, 1982). It is well-known that the uniform rule satisfies many desirable properties including strategy-proofness.² An earlier version of SSY points out that the direct revelation game of the uniform rule allows a continuum of inefficient Nash equilibria: the uniform rule is not securely implementable. Given this impossibility, we analyze which rules are securely implementable. Since we are considering a fair allocation problem, we require a rule to treat identical agents equally. This standard fairness requirement is called *symmetry*. In our first theorem, we show that there exists one and only one symmetric and securely implementable rule: it is the equal division rule. Since the equal division rule reflects no information on reported preferences and is inefficient, this result is rather negative. Also, since the uniform rule is symmetric, this result explains why the uniform rule fails to be securely implementable.

Theorem 1 exhibits how restrictive secure implementability is. However, given the central position of the uniform rule in the literature, we shall show a way out of the negative result. In our second main theorem, we show that any “good” Nash equilibrium is robust against coalitional deviations in the sense of strong Nash equilibrium. Moreover, we clarify how vulnerable any bad Nash equilibrium is to coalitional deviations. In our third main theorem, we show that (i) the true uniform allocation Pareto dominates all bad Nash equilibrium allocations, and (ii) at every bad Nash equilibrium, the coalition of agents who prefer the true uniform allocation to the Nash

¹This problem is observed in laboratory experiments of Clarke-Groves mechanisms by Cason, Saijo, Sjöström, Yamato (2006).

²See, for example, Sprumont (1991), Ching (1994), and Serizawa (2006) for characterizations of the uniform rule on the basis of strategy-proofness or its variants.

equilibrium allocation can credibly deviate from there to a strong Nash equilibrium that realizes the true uniform allocation. Furthermore, finding deviation strategies is quite easy: deviating agents simply need to report their true preferences. Thus, a Nash equilibrium is coalitionally stable if and only if it is “good”. Our results imply that the impossibility of securely implementing the uniform rule can be naturally resolved by allowing pre-play communication among players. Finally, a by-product of our results is that the uniform rule is triply implemented by its associated direct revelation mechanism in dominant strategy, coalition-proof Nash, and strong Nash equilibria.

The paper is organized as follows. In Section 2, we introduce the model, and we analyze secure implementation in Section 3. Next, we analyze the possibility of credible coalitional deviations in Section 4. In Section 5, we investigate coalition-proof Nash equilibrium. Finally, we provide some concluding comments in Section 6. All the proofs are relegated to the Appendix.

2 Definitions

Let $I \equiv \{1, 2, \dots, n\}$ be the finite set of *agents*. There is a fixed amount of an infinitely divisible resource $\Omega > 0$ to be allocated. An *allotment* for $i \in I$ is $x_i \in [0, \Omega]$. An *allocation* is a vector of allotments $x = (x_1, \dots, x_n) \in [0, \Omega]^I$ such that $\sum_{i \in I} x_i = \Omega$. Let X be the set of allocations.

A *single-peaked preference* is a transitive, complete, and continuous binary relation R_i over $[0, \Omega]$ for which there exists a “peak” $r_i \in [0, \Omega]$ such that, for each $x_i, x'_i \in [0, \Omega]$,

$$\begin{aligned} x'_i < x_i \leq r_i &\implies x_i P_i x'_i, \\ r_i \leq x_i < x'_i &\implies x_i P_i x'_i. \end{aligned}$$

The symmetric and asymmetric parts of R_i are denoted by I_i and P_i , respectively. Let D be the set of single-peaked preferences. A preference profile is $R \equiv (R_i)_{i \in I} \in D^I$ and the *peak profile* of $R \in D^I$ is $r \equiv (r_i)_{i \in I} \in [0, \Omega]^I$.

Given $T \subseteq I$, $R_T \in D^T$ and allocations $x, y \in X$, we write x wdom $[R_T]$ y if,

$$\begin{aligned} x_i R_i y_i &\text{ for each } i \in T, \\ x_j P_j y_j &\text{ for some } j \in T. \end{aligned}$$

Then x is said to *weakly dominate* y at R_T . Similarly, we write x sdom $[R_T]$ y if,

$$x_i P_i y_i \text{ for each } i \in T.$$

Then x is said to *strictly dominate* y at R_T . Obviously, strict domination implies weak domination.

Given $R \in D^I$ and allocations $x, y \in X$, x *Pareto dominates* y at R if x wdom $[R]$ y . Allocation $x \in X$ is *(Pareto-)efficient* at R if there exists no $y \in Y$ that Pareto dominates x at R . Let $P(R) \subseteq X$ be the set of efficient allocations at R . It is easy to see that $x \in P(R)$ if and only if

$$\begin{aligned}\Omega \leq \sum_{j \in I} r_j &\implies [x_i \leq r_i \quad \text{for each } i \in I], \\ \Omega \geq \sum_{j \in I} r_j &\implies [x_i \geq r_i \quad \text{for each } i \in I].\end{aligned}$$

A *rule* is a function $\psi : D^I \rightarrow X$, which maps a preference profile $R \in D^I$ to an allocation $\psi(R) \in X$. The rule that has played the prominent role in the literature is the *uniform rule* (Benassy, 1982; Sprumont, 1991):

Uniform rule, U : for each $R \in D^I$ and each $i \in I$,

$$U_i(R) = \begin{cases} \min\{r_i, \lambda\} & \text{if } \sum_{j \in I} r_j \geq \Omega, \\ \max\{r_i, \lambda\} & \text{if } \sum_{j \in I} r_j \leq \Omega, \end{cases}$$

where λ solves $\sum_{j \in I} U_j(R) = \Omega$.

It is well-known that the uniform rule satisfies many desirable properties (see, Thomson (2005) for a survey).

3 Secure implementation

The notion of secure implementation is introduced in the seminal work of SSY. A rule is *strategy-proof* if any true preference profile constitutes a dominant strategy equilibrium in its associated direct revelation game. Furthermore, a strategy-proof rule is *secure* if every Nash equilibrium in the game realizes the same outcome as the outcome obtained under true preferences.

Strategy-proofness: For each $R \in D^I$, each $i \in I$, and each $R'_i \in D$, $\psi_i(R) R_i \psi_i(R'_i, R_{-i})$.

Given a (true) preference profile $R^0 \in D^I$, a (reported) preference profile $R \in D^I$ is a *Nash equilibrium in the direct revelation game of ψ at R^0* if for each $i \in I$ and each $R'_i \in D$, $\psi_i(R) R_i^0 \psi_i(R'_i, R_{-i})$. Hereafter, the direct revelation game of ψ is simply referred to as the *game of ψ* . Let $N(\psi, R)$ be the set of Nash equilibria in the game of ψ at R .

Secure implementability: ψ is strategy-proof and for each $R^0 \in D^I$ and each $R \in N(\psi, R^0)$, $\psi(R) = \psi(R^0)$.³

An earlier version of SSY points out that the uniform rule is not securely implementable. The underlying reason is that its direct revelation mechanism allows for some “bad” Nash equilibria.⁴ For example, consider the two person case where $R^0 \in D^I$ is such that $r_1^0 < 0.5 < r_2^0$ and $\Omega = 1 < r_1^0 + r_2^0$. Then R with $r = (0.5, 0.5)$ is a Nash equilibrium but $U(R) = (0.5, 0.5) \neq U(R^0)$. Indeed, $U(R)$ is not even efficient at R^0 .

We first investigate the existence of securely implementable rules under the following standard fairness requirement, which states the symmetric treatment of two identical agents:

Symmetry: For each $R \in D^I$ and each $i, j \in I$ such that $R_i = R_j$, $\psi_i(R) = \psi_j(R)$.

A rule ψ that trivially satisfies symmetry and secure implementability is the invariant rule that always recommends the point of equal division, i.e. for each $i \in I$ and each $R \in D^I$, $\psi(R) = \frac{\Omega}{n}$. We call this rule the *equal division rule*. Theorem 1 below shows that there is no symmetric and securely implementable rule other than the equal division rule:

Theorem 1. *A rule is symmetric and securely implementable if and only if it is the equal division rule.*

Proof. See, the Appendix. □

Theorem 1 is rather disappointing, since the equal division rule reflects no information on reported preferences and is inefficient. Given that the uniform rule is symmetric, it also explains why this rule fails to be securely implementable.

In the problem of allocating a divisible resource where preferences are fixed to be strictly increasing and agents are characterized by certain non-preference parameters, Mizukami, Saijo, and Wakayama (2005) show that the equal division rule is the only symmetric and strategy-proof rule. In our problem, it is important to notice that the pair of symmetry and strategy-proofness does not characterize the equal division rule, since the uniform rule satisfies the two properties.⁵

³The original definition of secure implementability by SSY admits implementation by any indirect mechanism. However, the revelation principle of secure implementation (SSY, Theorem 1, Lemma 2) allows us to restrict our attention to the direct mechanism.

⁴The RIETI Discussion Paper (September 2003-E-019) version of SSY offers a discussion on the uniform rule. However, the latest version of SSY contains none of them.

⁵There are many other rules that satisfy symmetry and strategy-proofness. For instance, the rule

4 Coalitional stability of Nash equilibria

The difficulty of secure implementation observed in Theorem 1 motivates us to analyze whether bad Nash can be naturally eliminated by certain plausible criteria of equilibrium selection. In particular, given that the uniform rule has occupied a prominent role in the literature, we shall show a way out of the negative result. Our approach is to distinguish good and bad Nash equilibria in views of coalitional stability.

A profile $R \in D^I$ is a *strong Nash equilibrium* (Aumann, 1959) in the game of ψ at R^0 if there exist no $T \subseteq I$ and $R'_T \in D^T$ such that $\psi(R'_T, R_{I \setminus T}) \text{ wdom}[R_T^0] \psi(R)$.⁶ Let $SN(\psi, R^0)$ be the set of strong Nash equilibria in the game of ψ at R^0 . The next theorem ensures strong coalitional stability of any good Nash equilibrium.

Theorem 2. *For each $R^0 \in D^I$ and each $R \in N(U, R^0)$, if $U(R) = U(R^0)$, then $R \in SN(U, R^0)$.*

Proof. See, the Appendix. □

We remark that Theorem 2 generalizes the known fact that the uniform rule is “coalition strategy-proof”, which states that the profile of true preferences constitutes a strong Nash equilibrium in its associated direct revelation game (e.g., Serizawa, 2006).

Given that any good Nash equilibrium is coalitionally stable, if any bad Nash equilibrium is shown to be unstable in some sense, then we can distinguish good and bad Nash equilibria in views of coalitional stability and can conclude that only good Nash equilibria survive the stability test. Before establishing this implication, we first introduce a notion of credible coalitional deviations.

Given $R^0 \in D^I$ and $R \in N(\psi, R^0)$ such that $U(R) \neq U(R^0)$, a coalition $T \subseteq I$ can *robustly deviate from R via $R'_T \in D^T$ at R^0* if “the deviation is strongly profitable”:

$$U(R'_T, R_{I \setminus T}) \text{ sdom}[R_T^0] U(R),$$

and “no weakly profitable counter deviation is possible”: there exist no $S \subseteq I$ and $R''_S \in D^S$ such that

$$U(R''_S, R'_{T \setminus S}, R_{I \setminus (T \cup S)}) \text{ wdom}[R_S^0] U(R'_T, R_{I \setminus T}).$$

Strong deviation is quite a demanding condition on credible deviations, and hence we can conclude that any Nash equilibrium blocked by such a deviation is unrealizable.

which recommends the point of equal division whenever the peak of agent 1 is less than $\frac{\Omega}{n}$, and follows the uniform rule otherwise is symmetric and strategy-proof as well.

⁶We are defining strong Nash by weak domination, but all of our results hold if it is defined by strong domination.

Our next result shows that for every bad Nash equilibrium in the game of the uniform rule, (i) the uniform allocation Pareto dominates the bad Nash equilibrium allocation, (ii) agents who prefer the uniform allocation to the bad Nash equilibrium allocation are “misreporting” agents, (iii) the coalition of such misreporting agents can robustly deviate from the bad Nash equilibrium via their true preferences, (iv) the robust deviation leads to the uniform allocation, and (v) the robust deviation does not change the allotment of anyone who does not belong to the coalition. The following example illustrates this result for a three-person case.

Example 1. There are three agents and the preference profile R^0 is such that $r^0 = (1, 2, 4)$. The total resource available is $\Omega = 6$. Observe that R with $r = (2, 2, 2)$ and $U(R) = (2, 2, 2)$ is a Nash equilibrium in the game of the uniform rule at R^0 . By definition of the uniform rule, no one can change this allocation by any unilateral deviation from r . It is crucial here that the misreport is such that $r_1 + r_2 + r_3 = \Omega$. Notice that $U(R^0) = (1, 2, 3)$ Pareto dominates $U(R)$ at R^0 . Agents 1 and 3 have the joint profitable deviation of simply reporting their true preferences so that the true uniform allocation is obtained. By the same token, observe that the report R' with $r = (1.5, 2, 2.5)$ and $U(R') = (1.5, 2, 2.5)$ is also a Nash equilibrium in the game of the uniform rule at R^0 . This turns to be a general observation. Bad Nash equilibria occur only when the sum of the reported peaks is equal to Ω and the obtained allocation is Pareto dominated by the true uniform allocation. ♦

Theorem 3. For each $R^0 \in D^I$ and each $R \in N(U, R^0)$, if $U(R) \neq U(R^0)$, then the following statements hold:

(i) $U(R^0) \text{ wdom}[R^0] U(R)$, and so

$$T \equiv \{i \in I : U_i(R^0) P_i^0 U_i(R)\} \neq \emptyset;$$

(ii) $r_i \neq r_i^0 \quad \forall i \in T$;

(iii) T can robustly deviate from R via R_T^0 at R^0 ;

(iv) $U(R_T^0, R_{I \setminus T}) = U(R^0)$;

(v) $U_i(R_T^0, R_{I \setminus T}) = U_i(R) \quad \forall i \in I \setminus T$.

Proof. See, the Appendix. □

In a companion paper, Bochet and Sakai (2007) show that in the game of the uniform rule, any efficient Nash equilibrium allocation is the uniform allocation. Theorem 3 (i) generalizes this result, since the fact that the uniform allocation Pareto dominates a bad Nash equilibrium allocation implies that the bad Nash equilibrium allocation is not efficient.

Note that, in the robust deviation in Theorem 3, finding deviation strategies and calculating the final outcome are quite easy: deviating agents only need to report their true preferences so as to realize the true uniform allocation. Now the implication of Theorem 3 is clear: in the direct revelation mechanism associated with the uniform rule, all bad Nash equilibria can be naturally eliminated by allowing pre-play communication among players.

5 Coalition-proof Nash equilibrium

Bernheim, Peleg, and Winston (1987) introduce the notion of credibility of coalitional deviations and define an equilibrium as a position at which no credible coalitional deviation occurs. We define this notion based on two credibility conditions.

Coalition-proof Nash equilibrium under weak domination: We first inductively define the notion of *credible deviation under weak domination* in the game of ψ at R^0 : (1) *Singleton coalitions:* $T = \{i\}$ for some $i \in I$. In this case, $\{i\}$ can credibly deviate from R via R'_i if $\psi_i(R'_i, R_{-i}) \succ_i^0 \psi_i(R)$; (2) *Other coalitions:* any T with $|T| \geq 2$. In this case, T can credibly deviate from R via R'_T under weak domination if $\psi(R'_T, R_{I \setminus T}) \text{ wdom}[R_T] \psi(R)$ and there exist no $S \subseteq T$ and $R''_S \in D^S$ such that S can credibly deviate from $(R'_T, R_{I \setminus T})$ via R''_S under weak domination.

Then $R \in D^I$ is a *coalition-proof Nash equilibrium* if there exist no $T \subseteq I$ and $R'_T \in D^T$ such that T can credibly deviate from R via R'_T under weak domination in the game of ψ at R^0 . Let $CN_w(\psi, R^0)$ be the set of coalition-proof Nash equilibria under weak domination in the game of ψ at R^0 .

Coalition-proof Nash equilibrium under strong domination: This notion is defined by replacing “weak domination” with “strong domination” in the definitions of credible deviation under weak domination and coalition-proof Nash equilibrium under weak domination. Let $CN_s(\psi, R^0)$ be the set of coalition-proof Nash equilibria under strong domination in the game of ψ at R^0 .

By definition, any strong Nash equilibrium is a coalition-proof Nash equilibrium under weak domination and under strong deviation, but the converse does not always hold. Indeed, coalition-proof Nash equilibria can be even inefficient. Furthermore, Konishi, Le Breton, and Weber (1999) point out that the set of coalition-proof Nash equilibria under weak domination and the set under strong domination may have an empty intersection. However, in the game of the uniform rule, both sets coincide and each element of the sets realizes the uniform allocation. This is because every bad Nash equilibrium can be blocked by a coalitional deviation that meets both credibility

conditions (Theorem 3), while every good Nash equilibrium is a strong Nash equilibrium (Theorem 2). The next theorem summarizes this fact and other characteristics of “good” Nash equilibria:

Theorem 4. *For each $R^0 \in D^I$ and each $R \in N(U, R^0)$, the following statements are equivalent:*

- (i) $U(R) = U(R^0)$;
- (ii) $U(R)$ is efficient at R^0 ;
- (iii) There exists no $R' \in N(U, R^0)$ such that $U(R') \text{ wdom}[R^0] U(R)$;
- (iv) $R \in CN_w(U, R^0)$;
- (v) $R \in CN_s(U, R^0)$;
- (vi) $R \in SN(U, R^0)$.

Proof. By Theorem 2, (i) implies (vi). By definition, (vi) implies (ii, iii, iv, v). By Theorem 3, each one of (ii, iii, iv, v) implies (i). \square

A few remarks are in order. Shinohara (2005) shows that, in a class of games in which each player’s payoff only depends on his own strategy and the sum of other players’ strategies, the set of coalition-proof Nash equilibria under weak domination is contained by the set under strict domination. Obviously, the game of the uniform rule does not belong to the class, since each player’s payoff depends on the entire distributions of peaks. Hence our equivalence between (iv) and (v) is independent of Shinohara’s observation.

Note that (iii) implies the equivalence between the Pareto frontier of the set of Nash equilibria and the set of coalition-proof Nash equilibria (under any dominance relation). Yi (1999) establishes the same equivalence in games with strategic substitutes where each player’s payoff depends only on his own strategy and the sum of others’ strategies. However, the game of the uniform rule apparently differs from those games, so our equivalence between the two sets is independent of Yi’s result.

In terms of implementation theory, (iv, v, vi) imply that the direct revelation mechanism of the uniform rule implements the uniform rule in coalition-proof Nash (under both dominance relations) and strong Nash equilibria.

6 Concluding comments

We studied secure implementation in the problem of fairly allocating a divisible resource when agents have single-peaked preferences. We first showed that the equal division rule is the only rule that is symmetric and securely implementable. To overcome this impossibility, we analyzed the structure of bad Nash equilibria in the game

of the uniform rule and showed that they can be naturally eliminated by a credible coalitional deviation that realizes the true uniform allocation. This significantly contrasts with the robustness of good Nash equilibria to arbitrary coalitional deviations. Thus the impossibility of securely implementing the uniform rule can be resolved when pre-play communication among players is possible.

In this paper, we only focused on symmetric rules. However, there are interesting classes of efficient and strategy-proof rules that are asymmetric and respect certain priority over agents (e.g., Barberà, Jackson, and Neme, 1997; Moulin, 1999). It seems interesting to investigate which rules in these classes are securely implementable. We leave this question open for future research.

Appendix: Proofs

Proof of Theorem 1

The proof proceeds in several lemmas. In the proof, we clarify logical relations among secure implementability and other standard properties that might be of independent interest.

Characterizations of secure implementability by SSY imply that the pair of strategy-proofness and the following independence condition is necessary and sufficient for secure implementation:

Rectangular property: For each $R, R' \in D^I$, if for each $i \in I$, $\psi_i(R) \succeq_i \psi_i(R'_i, R_{-i})$, then $\psi(R) = \psi(R')$.

Lemma 1. *A rule is securely implementable if and only if it satisfies strategy-proofness and the rectangular property.*

Proof. See, Theorem 1 in SSY. □

The next condition is an informational restriction on the choice of allocations. It states that a rule determines an allocation only depending on the peaks of preferences:

Peak-only: For each $R, R' \in D^I$, if $r = r'$, then $\psi(R) = \psi(R')$.

Lemma 2. *If a rule is securely implementable, then it satisfies peak-only.*

Proof. Let ψ be a securely implementable rule. By Lemma 1, ψ satisfies strategy-proofness and the rectangular property. It suffices to show that, for each $R \in D^I$, $i \in I$, and $R'_i \in D$ with $r_i = r'_i$, $\psi(R) = \psi(R'_i, R_{-i})$.

Let $x \equiv \psi(R)$ and $y \equiv \psi(R'_i, R_{-i})$. Without loss of generality, assume that $x_i \leq y_i$. By strategy-proofness at R , it is not possible that $x_i < y_i \leq r_i$. By strategy-proofness

at (R'_i, R_{-i}) , it is not possible that $r'_i \leq x_i < y_i$. Hence, $x_i \leq r_i \leq y_i$. Let $R''_i \in D$ be such that $x_i I''_i y_i$ and $r''_i = r_i$.

Let $z \equiv \psi(R''_i, R_{-i})$. By strategy-proofness, either $x_i = z_i$ or $y_i = z_i$. Consider the case $x_i = z_i$. By the rectangular property, $x = z$. Note that

$$\psi_i(R''_i, R_{-i}) = z_i I''_i y_i = \psi_i(R'_i, R_{-i}).$$

By the rectangular property, $y = z$. Hence $x = y$. The same logic applies to the case $y_i = z_i$. \square

The next condition, introduced by Satterthwaite and Sonnenschein (1981), states that no one can change anyone else's allotment unless he changes his own:

Non-bossiness: For each $R \in D^I$, each $i \in I$, and each $R'_i \in D$, if $\psi_i(R) = \psi_i(R'_i, R_{-i})$, then $\psi(R'_i, R_{-i}) = \psi(R)$.

Obviously, the rectangular property implies non-bossiness.

The next condition states that any chosen allocation should be such that no one prefers someone else's allotment to his own (Foley, 1967):

Envy-freeness: For each $R \in D^I$ and each $i, j \in I$, $\psi_i(R) R_i \psi_j(R)$.

Lemma 3. *If a rule is symmetric, strategy-proof, and non-bossy, then it is envy-free.*

Proof. This is implied by Theorem 2 in Fleurbaey and Maniquet (1997). \square

Lemma 4. *If a rule satisfies the rectangular property, peak-only, and envy-freeness, then it is the equal division rule.*

Proof. Let ψ be a rule satisfying the three properties. Let \bar{R} be such that for each $i \in I$, $\bar{r}_i = \frac{\Omega}{n}$. Let $x \equiv \psi(\bar{R})$.

Claim 1: For each $i \in I$, $x_i = \frac{\Omega}{n}$. Suppose, by contradiction, that there exist $j, h \in I$ such that

$$x_j < \frac{\Omega}{n} < x_h.$$

Let R'_j be such that $r'_j = \frac{\Omega}{n}$ and

$$x_h P'_j x_j.$$

By peak-only, $x = \psi(R'_j, \bar{R}_{-j})$. But then j envies h , a contradiction with envy-freeness.

Claim 2: For each $R \in D^I$, $\psi(R) = x$. Let $R \in D^I$, $i \in I$ and $y \equiv \psi(R_i, \bar{R}_{-i})$. Our purpose is to show that $y_i = \frac{\Omega}{n}$. We shall derive a contradiction for the case $y_i < \frac{\Omega}{n}$. The opposite case can be dealt with in a parallel way, so we omit it.

Since $y_i < \frac{\Omega}{n}$, there exists $j \neq i$ such that $\frac{\Omega}{n} < y_j$. Let R'_j be such that $r'_j = \frac{\Omega}{n}$ and

$$y_i P'_j y_j.$$

By peak-only, $y = \psi(R'_j, R_i, \bar{R}_{-ij})$. But then j envies i , a contradiction to envy-freeness. Therefore, $y_i = \frac{\Omega}{n}$.

We showed that for each $i \in I$,

$$\psi_i(R_i, \bar{R}_{-i}) \bar{I}_i \psi_i(\bar{R}).$$

Thus by the rectangular property, $\psi(R) = \psi(\bar{R})$. That is, $\psi(R)$ is the equal division allocation. \square

Proof of Theorem 1. Obviously, the equal division rule is symmetric and securely implementable.

Conversely, if a rule is symmetric and securely implementable, then by Lemmas 1, 2, and 3, it also satisfies the rectangular property, peak-only, and envy-freeness. Hence by Lemma 4, the rule is the equal division rule. \square

Proof of Theorem 2

We fix the true preference profile R^0 throughout the proof of Theorem 2. We only deal with the case $\Omega \leq \sum_{i \in I} r_i^0$. The opposite case can be handled in a similar way.

Lemma 5. *There exists no $R \in N(U, R^0)$ with $\sum_{i \in I} r_i < \Omega$.*

Proof. Suppose not. There exists $R \in N(U, R^0)$ such that $\sum_{i \in I} r_i < \Omega$. Then there exists $\lambda \in (0, \frac{\Omega}{n})$ such that

$$x_i = \max\{r_i, \lambda\} \quad \forall i \in I.$$

Observe that every $i \in I$ could get more by reporting a preference whose peak is more than $\max\{r_i, \lambda\}$. Therefore,

$$r_i^0 \leq \psi_i(R) \quad \forall i \in I.$$

But this is a contradiction with $\Omega \leq \sum_{i \in I} r_i^0$. \square

In Lemmas 6 and 7, we write

$$I_1 \equiv \{i \in I : x_i = r_i^0 < \lambda\}, \tag{1}$$

$$I_2 \equiv \{i \in I : x_i = \lambda \leq r_i^0\}, \tag{2}$$

where x, λ are such that $x = U(R^0) = (\min\{r_i^0, \lambda\})_{i \in I}$. Note that I_1, I_2, x, λ depend on R^0 , but we use these notations for simplicity.

Lemma 6. For each $R \in N(U, R^0)$, if $U(R) = U(R^0) \neq r^0$ and $\Omega \leq \sum_{i \in I} r_i$, then

$$\begin{aligned} r_i &= r_i^0 \quad \forall i \in I_1, \\ \lambda &\leq r_i \quad \forall i \in I_2. \end{aligned}$$

Proof. Let $R \in N(U, R^0)$. Assume that $U(R) = U(R^0) \neq r^0$ and $\Omega \leq \sum_{i \in I} r_i$.

Consider the case $\Omega = \sum_{i \in I} r_i$. Then it follows that $x = r$. Hence, for each $i \in I_1$, $r_i = x_i = r_i^0$, and for each $i \in I_2$, $r_i = x_i = \lambda$.

Consider the case $\Omega < \sum_{i \in I} r_i$. Then for each $i \in I_1$, since $U_i(R) = r_i^0$, we get that $r_i = r_i^0$. Obviously, for each $i \in I_2$, $\lambda = x_i \leq r_i$. \square

Lemma 7. For each $R \in N(U, R^0)$, if $U(R) = U(R^0) \neq r^0$, $\Omega \leq \sum_{i \in I} r_i$, and

$$\begin{aligned} r_i &= r_i^0 \quad \forall i \in I_1, \\ \lambda &\leq r_i \quad \forall i \in I_2, \end{aligned}$$

then there exist no $T \subseteq I$ and $R'_T \in D^T$ such that

$$U(R'_T, R_{I \setminus T}) \text{ wdom}[R_T^0] U(R).$$

Proof. Suppose, by contradiction, that there exist $T \subseteq I$ and $R'_T \in D^T$ such that

$$U(R'_T, R_{I \setminus T}) \text{ wdom}[R_T^0] U(R).$$

Let $y \equiv U(R'_T, R_{I \setminus T})$ and $j \in T$ be such that $y_j P_j^0 x_j$. Then $j \in I_2$ and $x_j < y_j$.

We first consider the case $\sum_{i \in T} r'_i + \sum_{i \in I \setminus T} r_i \geq \Omega$. Let $\mu \in [\frac{\Omega}{n}, \Omega]$ be such that $y = ((\min\{r'_i, \mu\})_{i \in T}, (\min\{r_i, \mu\})_{i \in I \setminus T})$. Note that $x_j < y_j$ implies $\lambda < \mu$. Then for each $i \in I \setminus T$, $x_i \leq y_i$. Thus there exists $k \in T$ such that $y_k < x_k$, a contradiction.

We next consider the case $\sum_{i \in T} r'_i + \sum_{i \in I \setminus T} r_i < \Omega$. Let $\mu \in [0, \frac{\Omega}{n}]$ be such that $y = ((\max\{r'_i, \mu\})_{i \in T}, (\max\{r_i, \mu\})_{i \in I \setminus T})$. For each $i \in I \setminus T$, since $y_i = \max\{r_i, \mu\}$ and $x_i = r_i$, we have $x_i \leq y_i$. Hence there exists $k \in T$ such that $y_k < x_k$, a contradiction. \square

Lemma 8. For each $R \in N(U, R^0)$, if $U(R) = U(R^0) \neq r^0$ and $\Omega \leq \sum_{i \in I} r_i$, then there exist no $T \subseteq I$ and $R'_T \in D^T$ such that

$$U(R'_T, R_{I \setminus T}) \text{ wdom}[R_T^0] U(R).$$

Proof. Follows from Lemmas 6 and 7. \square

Proof of Theorem 2. Let $R \in N(U, R^0)$ be such that $U(R) = U(R^0)$. If $U(R) = r^0$, then obviously no coalition can be weakly made better off by any deviation from R . Hence consider the case $U(R^0) \neq r^0$. By Lemma 5, $\Omega \leq \sum_{i \in I} r_i$. Thus by Lemma 8, $R \in SN(U, R^0)$. \square

Proof of Theorem 3

Again, we fix the true preference profile R^0 throughout the proof of Theorem 3. We only deal with the case $\Omega \leq \sum_{i \in I} r_i^0$. The opposite case can be handled in a similar way.

Lemma 9. *For each $R \in N(U, R^0)$ such that $U(R) \neq U(R^0)$, if $U(R) = (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$, then the conclusion of Theorem 3 holds.*

Proof. Let $R \in N(U, R^0)$ be such that $U(R) \neq U(R^0)$. Assume that $x \equiv U(R) = (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$.

Case 1: $\Omega < \sum_{i \in I} r_i$. Let $\lambda > \frac{\Omega}{n}$ be such that $x = (\min\{\lambda, r_i\})_{i \in I}$. Since $R \in N(U, R^0)$, it is easy to see that for each $i \in I$, $x_i \leq r_i^0$. Then one can easily verify that

$$x_i = \min\{\lambda, r_i^0\} \quad \forall i \in I.$$

Hence $x = U(R^0)$, which contradicts $U(R) \neq U(R^0)$. Thus this case does not occur.

Case 2: $\sum_{i \in I} r_i < \Omega$. Since $R \in N(U, R^0)$, it is easy to see that for each $i \in I$, $r_i^0 \leq x_i$. This is possible only if for each $i \in I$, $r_i^0 = x_i$. Hence $x = U(R^0)$, which contradicts $U(R) \neq U(R^0)$. Thus this case does not occur, too.

Case 3: $\Omega = \sum_{i \in I} r_i$. By Cases 1 and 2, only this case can occur. Let $\lambda \geq \frac{\Omega}{n}$ be such that $U(R^0) = (\min\{r_i^0, \lambda\})_{i \in I}$. Since $U(R)$ is the equal division but not the uniform allocation, $\lambda > \frac{\Omega}{n}$ and $U(R^0)$ Pareto dominates $U(R)$ at R^0 .

Let $T \equiv \{i \in I : r_i \neq r_i^0\}$ be the set of agents who are not reporting their true peaks. Because $U(R)$ is the equal division by assumption, we have that,

$$r_i = r_i^0 \implies r_i^0 = \frac{\Omega}{n} \quad \forall i \in I \setminus T, \quad (3)$$

$$r_i \neq r_i^0 \implies r_i^0 \neq \frac{\Omega}{n} \quad \forall i \in T. \quad (4)$$

(3) and (4) imply that, when coalition T deviates from R via R_T^0 , then the true uniform allocation $U(R^0)$ is obtained. Hence

$$U_i(R) = \frac{\Omega}{n} < \lambda = U_i(R^0) \leq r_i^0 \quad \forall i \in T \text{ with } \lambda < r_i^0,$$

$$U_i(R^0) = r_i^0 \quad \forall i \in T \text{ with } r_i^0 \leq \lambda.$$

Therefore $U(R_T^0, R_{I \setminus T}) \text{ sdom}[R_T^0] U(R)$. Robustness of this deviation is ensured by Lemma 6. Thus we verified all the properties of Theorem 3. \square

Lemma 10. For each $R \in N(U, R^0)$, if $\Omega < \sum_{i \in I} r_i$ and $U(R) \neq (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$, then $U(R) = U(R^0)$.

Proof. Let $R \in N(U, R^0)$ be such that $\Omega < \sum_{i \in I} r_i$ and $x \equiv U(R) \neq (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$.

Then there exists a number $\lambda \in (\frac{\Omega}{n}, \Omega)$ such that

$$x_i = \min\{r_i, \lambda\} \quad \forall i \in I.$$

Note that

$$I_1 \equiv \{i \in I : x_i = r_i < \lambda\} \neq \emptyset,$$

$$I_2 \equiv \{i \in I : x_i = \lambda \leq r_i\} \neq \emptyset,$$

$$I_1 \cap I_2 = \emptyset \text{ and } I_1 \cup I_2 = I.$$

Every $i \in I$ with $x_i > 0$ could get less by reporting a preference whose peak is less than $\min\{r_i, \lambda\}$. Therefore,

$$x_i \leq r_i^0 \quad \forall i \in I \text{ with } 0 < x_i. \quad (5)$$

In particular,

$$x_i \leq r_i^0 \quad \forall i \in I_2. \quad (6)$$

Every $i \in I_1$ with $x_i = 0$ could get more by reporting a preference whose peak is positive. Therefore,

$$x_i = r_i^0 \quad \forall i \in I_1 \text{ with } x_i = 0. \quad (7)$$

Furthermore, every $i \in I_1$ with $x_i > 0$ could get more by reporting a preference whose peak is more than r_i . This and (5) together imply that,

$$x_i = r_i^0 \quad \forall i \in I_1 \text{ with } x_i > 0. \quad (8)$$

By (6), (7), and (8),

$$x_i = \min\{r_i^0, \lambda\} \quad \forall i \in I.$$

Thus $x = U(R^0)$. □

Lemma 11. For each $R \in N(U, R^0)$ such that $U(R) \neq U(R^0)$, if $U(R) \neq (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$, then the conclusion of Theorem 3 holds.

Proof. Let $R \in N(U, R^0)$ be such that $U(R) \neq U(R^0)$. Assume that $U(R) \neq (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$. By $U(R) \neq U(R^0)$ and Lemma 5, $\Omega \leq \sum_{i \in I} r_i$. Therefore by $U(R) \neq (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$ and Lemma 10, $\sum_{i \in I} r_i = \Omega$.

Without loss of generality, assume that

$$x_1 \leq x_2 \leq \dots \leq x_n.$$

Since $\Omega = \sum_{i \in I} r_i$,

$$x_i = r_i \quad \forall i \in I. \quad (9)$$

Let $I_1 \equiv \{i \in I : x_i = x_1\}$ and $I_n \equiv \{i \in I : x_i = x_n\}$. Since $U(R) \neq (\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$, we have $x_1 < x_n$, and so $I_1 \cap I_n = \emptyset$.

Step 1: For each $i \in I$, if $x_1 < x_i$, then $r_i = x_i \leq r_i^0$. Then by definition of the uniform rule and $\Omega = \sum_{i \in I} r_i$, i could get less by reporting a preference whose peak is less than x_i . Hence $r_i = x_i \leq r_i^0$.

Step 2: For each $i \in I$, if $x_i < x_n$, then $r_i^0 \leq x_i = r_i$. Then by definition of the uniform rule and $\Omega = \sum_{i \in I} r_i$, i could get more by reporting a preference whose peak is more than x_i . Hence $r_i^0 \leq x_i = r_i$.

Step 3: For each $i \in I$, if $x_1 < x_i < x_n$, then $r_i = r_i^0 = x_i$. This is implied by Steps 1 and 2.

Step 4: Finding a credible deviation. By Steps 1 and 2,

$$\max_{i \in I_1} r_i^0 \leq x_1 \leq r_2^0 = x_2 \leq r_3^0 = x_3 \leq \dots \leq r_{n-1}^0 = x_{n-1} \leq x_n \leq \min_{i \in I_n} r_i^0. \quad (10)$$

If for each $i \in I_1$, $r_i^0 = x_i$, then $x = (\min\{r_i^0, x_n\})_{i \in I} = U(R^0)$, a contradiction. Hence $I'_1 \equiv \{i \in I_1 : r_i^0 < x_i\} \neq \emptyset$. This fact, the assumption $\Omega \leq \sum_{i \in I} r_i^0$, and (10) in turn imply that $I'_n \equiv \{i \in I_n : x_i < r_i^0\} \neq \emptyset$.

Let $T \equiv \{i \in I : r_i \neq r_i^0\}$ be the set of agents who are not reporting their true peaks. By the above discussion, $T = I'_1 \cup I'_n$. We consider the coalitional deviation of T at R via R_T^0 , which realizes the true uniform allocation $U(R_T^0, R_{I \setminus T}) = U(R^0)$.

We shall show that

$$U(R^0) \text{ sdom}[R_T^0] U(R).$$

Let $y \equiv U(R^0)$ and $\eta > \frac{\Omega}{n}$ be such that

$$y_i = \min\{r_i^0, \eta\} \quad \forall i \in I.$$

For each $i \in I'_1$, since $r_i^0 \leq \frac{\Omega}{n}$, $y_i = r_i^0 < x_i$. Thus

$$U(R^0) \text{ sdom}[R_{I'_1}^0] U(R). \quad (11)$$

We observed that, after the deviation by T , the allotment of every $i \in I'_1$ decreased from x_i to r_i^0 . This fact, (10), and $\sum_{i \in I} y_i = \Omega$ together imply that $x_n < \eta$. Also, $\Omega \leq \sum_{i \in I} r_i^0$ implies that $\eta \leq \max_{i \in I} r_i^0$. Overall, $x_n < \eta \leq \max_{i \in I} r_i^0$. Therefore,

$$\begin{aligned} x_i < y_i = r_i^0 \quad \forall i \in I'_n \text{ with } r_i^0 \leq \eta, \\ x_i < \eta = y_i < r_i^0 \quad \forall i \in I'_n \text{ with } \eta < r_i^0. \end{aligned}$$

Thus

$$U(R^0) \text{ sdom}[R_{I'_n}^0] U(R). \quad (12)$$

By (11, 12),

$$U(R^0) \text{ sdom}[R_T^0] U(R).$$

Robustness of this deviation is ensured by Lemma 6. Thus we verified all the properties of Theorem 3. \square

Proof of Theorem 3. Follows from Lemmas 9 and 11. \square

References

- Aumann, R.J. (1959) "Acceptable Points in General Cooperative n -person Games," in *Contribution to the Theory of Games IV*, Princeton University Press.
- Barberà, S., Jackson, M.O., and Neme, A. (1997) "Strategy-Proof Allotment Rules," *Games and Economic Behavior*, 18, 1-21.
- Benassy, J.-P. (1982) *The Economics of Market Disequilibrium*, New York Academic Press.
- Bernheim, D., Peleg, B., and Winston, M. (1987) "Coalition-Proof Nash Equilibria I. Concepts," *Journal of Economic Theory*, 42, 1-12.
- Bochet, O. and Sakai, T. (2007) "Preference Manipulations Lead to the Uniform Rule," mimeo, Maastricht University and Yokohama National University.
- Cason, T.N., Saijo, T., Sjöström, T., and Yamato, T. (2006) "Secure Implementation Experiments: Do Strategy-Proof Mechanisms Really Work?," *Games and Economic Behavior*, available on-line.

- Ching, S. (1994) “An Alternative Characterization of the Uniform Rule,” *Social Choice and Welfare*, 40, 57-60.
- Fleurbaey, M. and Maniquet, F. (1997) “Implementability and Horizontal Equity Imply No-Envy,” *Econometrica*, 65, 1215-1220.
- Foley, D. (1967) “Resource Allocation and the Public Sector,” *Yale Economic Essays*, 7, 45-98.
- Jackson, M.O. (2001) “A Crash Course in Implementation Theory,” *Social Choice and Welfare*, 18, 655-708.
- Maskin, E. (1999) “Nash Equilibrium and Welfare Optimality,” *Review of Economic Studies* 66, 23-38.
- Mizukami, H., Saijo, T., and Wakayama, T. (2005) “Strategy-Proof Sharing,” Discussion Papers in Economics and Business 05-05, Osaka University.
<http://www2.econ.osaka-u.ac.jp/library/global/dp/dp.html>
- Moulin, H. (1999) “Rationing a Commodity along a Fixed Path,” *Journal of Economic Theory*, 84, 41-72.
- Saijo, T., Sjöström, T., and Yamato, T. (2007) “Secure Implementation,” forthcoming in *Theoretical Economics*. RIETI Discussion Paper (03-E-019) version is available at
<http://www.rieti.go.jp/jp/publications/dp/03e019.pdf>.
- Satterthwaite, M. and Sonnenschein, H. (1981) “Strategy-Proof Allocation Mechanisms at Differentiable Points,” *Review of Economic Studies*, 48, 587-597.
- Serizawa, S. (2006) “Pairwise Strategy-Proofness and Self-Enforcing Manipulation,” *Social Choice and Welfare*, 26, 305-331.
- Shinohara, R. (2005) “Coalition-Proofness and Dominance Relations,” *Economics Letters*, 89, 174-179.
- Sprumont, Y. (1991) “The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule,” *Econometrica*, 59, 509-519.
- Thomson, W. (2005) *The Theory of Fair Allocation*, Ch. 11, unpublished book manuscript, University of Rochester.
- Yi, S.-S. (1999) “On the Coalition-Proofness of the Pareto Frontier of the Set of Nash Equilibria,” *Games and Economic Behavior*, 26, 353-364.