Preference manipulations lead to the uniform rule *

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Abstract

In the division problem with single-peaked preferences, it is well known that the uniform rule is robust to strategic manipulation. Furthermore, under efficiency and symmetry, it is the unique strategy-proof rule (Sprumont, 1991; Ching, 1994). We conversely analyze the consequences of strategic manipulation for a wide class of rules satisfying some mild conditions. Given a rule, we interpret its associated direct revelation game as a manipulation game, and we characterize its equilibrium allocations. We show that, for every rule that belongs to the class, there exists a unique strong Nash equilibrium allocation and it is the uniform allocation. Furthermore, it is also the unique Pareto-efficient Nash equilibrium allocation. Under an additional strict monotonicity condition, we show that the uniform allocation is the unique Nash equilibrium allocation. These results underline how strong the position of the uniform rule is in this model when the problem of strategic manipulation is an issue.

Keywords: Uniform rule, Degree of manipulability, Implementation, Existence of a strong Nash equilibrium, Double implementation, Secure implementation, Fair allocation.

JEL codes: C72, D63, D61, C78, D71.

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1 Introduction

This paper studies the problem of fairly allocating an amount of divisible resource to agents who have single-peaked preferences over the resource (Sprumont, 1991). The most prominent fact in this literature is the presence of the uniform rule (Benassy, 1982; Sprumont 1991), which is justified by many desirable properties. [1] In particular, the uniform rule is known to be quite robust to strategic manipulation. For instance, it is not only strategy-proof, but also coalitionally strategy-proof. Furthermore, under efficiency and symmetry, the uniform rule is the unique strategy-proof rule (Sprumont, 1991; Ching, 1994). However, while strategy-proofness is an important normative requirement, its violation does not clarify anything on the degree of manipulability of a rule. Thus, it is then unknown whether the consequences of strategic manipulation are “serious”. This argument motivates us to investigate strategic manipulation for a wide class of rules Ψ^1 satisfying some weak requirements of efficiency, monotonicity and continuity. It contains the uniform rule, the proportional rule and the equal loss rule among many others.[2] Indeed, almost all symmetric and continuous rules discussed in the literature belong to Ψ^1.

To analyze the consequence of strategic manipulation of a rule, we consider its associated direct revelation game (henceforth, game), and we interpret it as a manipulation game. [3] In the (manipulation) game, we consider that Nash equilibrium or its refinements are played. We first show that, in the game of any rule in Ψ^1, there exists one and only one strong Nash equilibrium allocation and it is the uniform allocation. Furthermore, this allocation is the unique Pareto efficient Nash equilibrium allocation. This result implies that each rule in Ψ^1 coincides with the uniform rule as a consequence of strategic manipulation via strong Nash equilibria.

The above result, however, does not uncover the complete structure of the set of Nash equilibrium allocations, though it fully characterizes the sets of Pareto efficient Nash and strong Nash equilibrium allocations. For several rules $\psi \in \Psi^1$, the game of ψ may admit Nash equilibrium allocations that differ from the uniform allocation. An example of such a rule is actually the uniform rule, as first pointed out by Saijo, Sjöström, and Yamato (2003). We next narrow down our class to $\Psi^2 \subsetneq \Psi^1$ by imposing an additional strict monotonicity condition. It turns out that, in the game of each $\psi \in \Psi^2$, there is no Nash equilibrium allocation other than the

[1] See, for example, Ching (1994), Thomson (1994a,b, 1995, 1997), among many others. Thomson (2005, Ch.11) offers a survey of the literature.

[2] Our results do not “strictly speaking” cover the proportional rule. We require a form of continuity that the proportional rule does not satisfy. Indeed, the proportional rule is discontinuous around 0. We comment on this issue in section 4.3

[3] There are some studies that analyze the consequence of strategic manipulation in different economic models. However, not so many efforts have been carried out to this approach, which much contrasts with the “standard" approach that seeks for implementable rules or mechanisms implementing rules. Nash-style equilibrium in manipulation games is first studied by Hurwicz (1978) in economies with divisible goods. Thomson (1988) offers a brief survey of the literature in earlier days. For recent studies on this topic, see Takamiya (2006, Sect 0.2) and its references for matching markets, Tadenuma and Thomson (1995) and Fujinaka and Sakai (2006a,b) for economies with indivisibilities.

uniform allocation. [4] In terms of implementation theory, this result implies that the game of each $\psi \in \Psi^2$ doubly implements the uniform rule in Nash and strong Nash equilibria. [5] The class Ψ^2 includes the proportional rule and the equal loss rule among many others, but not the uniform rule.

A corollary to our results is that the game of the uniform rule implements the uniform rule in strong Nash equilibrium but not in Nash equilibrium, while the game of any $\psi \in \Psi^2$ doubly implements the uniform rule in Nash and strong Nash equilibria. This suggests the use of those rules to realize the uniform allocation instead of the uniform rule itself when "bad" Nash equilibria in the game of the uniform rule matter. [6] Moreover, if agents behave according to the Nash equilibrium criterion –or its refinements– then the failure of the direct revelation game of the uniform rule is not an issue. A planner can substitute the uniform rule as outcome function by any rule in Ψ^2.

Our results underline how strong the position of the uniform rule is in this model when the problem of strategic manipulation is an issue. Manipulation of preferences in direct revelation games lead to recommendations made by the uniform rule. Thus, our results are both positive and negative. A positive aspect is that through strategic manipulation, the distributional objectives of the uniform rule are preserved, e.g. efficiency and envy-freeness. On the other hand, a negative aspect is that any distributional objective that the uniform allocation does not posses cannot be achieved.

Before proceeding, a clarification is in order. Dasgupta, Hammond and Maskin (1979) (DHM for short) show that in various economic environments, a single-valued rule is Maskin monotonic –a necessary condition for Nash implementation, (Maskin, 1999)– if and only if it is strategy-proof. A common misunderstanding is to evaluate our results as a corollary of theirs. Their result states that Nash implementability requires strategy-proofness, but this implies nothing on the set of Nash equilibrium allocations of the game of each of the rule in Ψ^2. In particular, given $\psi \in \Psi^2$, the set of Nash equilibrium allocations can be empty or multivalued, in which case DHM's result has no bite. [7] Moreover, even if the set of Nash equilibrium allocations is a singleton at each preference profile, its coincidence with the uniform rule is still not ensured. In fact, nothing guarantees that the properties of the rule used as

[4] An independent related work by Thomson (2004) shows that if a rule satisfies a condition similar to our strict monotonicity and a continuity property, then any Nash equilibrium allocation, if exists, is the uniform allocation. Thomson checks the existence of a Nash equilibrium for a few interesting rules, but does not offer any general existence result. On the other hand, our result involves the existence of a strong Nash equilibrium for a large class of rules. We also remark that he focus on Nash equilibrium and does not study strong Nash equilibria. The authors thank him for sending the paper and communication.

[5] Note that any rule that is doubly implementable in Nash and strong Nash equilibria is also implementable in any solution concept that is between these two equilibrium concepts. Pareto efficient Nash equilibrium and coalition-proof Nash equilibrium (Bernheim, Peleg, and Winston, 1987) are such examples. Double implementation in Nash and strong Nash equilibria is introduced by Maskin (1979) and its necessary and sufficient condition is obtained by Suh (1997).

[6] A recent paper, Bochet and Sakai (2006), focus on this topic in its relation to "secure implementation" introduced by Saijo, Sjöström, and Yamato (2003).

[7] Moreover, as it will made clear later, non-emptiness of the Nash equilibrium set is non-trivial.

outcome function are preserved through strategic manipulation: one can obtain a strategy-proof rule that differs from the uniform rule.[8]

The rest of the paper proceeds as follows. Section 2 presents the model and definitions. Section 3 presents our two main results. Section 4 discusses the results. Section 5 concludes the paper. All proofs are relegated to the Appendix.

2 Definitions

2.1 Basic definitions

Let $I \equiv \{1, 2, \ldots, n\}$ be the finite set of *agents*. There is a fixed amount of an infinitely divisible resource $\Omega > 0$ to be allocated. An *allotment* for $i \in I$ is $x_i \in [0, \Omega]$. An *allocation* is a vector of allotments $x = (x_1, \ldots, x_n) \in [0, \Omega]^I$ such that $\sum_{i \in I} x_i = \Omega$. Let X be the set of allocations.

A *single-peaked preference* is a transitive, complete, and continuous binary relation R_i over $[0, \Omega]$ for which there exists a "peak" $p_i \in [0, \Omega]$ such that, for each $x_i, x_i' \in [0, \Omega]$,

$$x_i' < x_i \leq p_i \Longrightarrow x_i \; P_i \; x_i',$$
$$p_i \leq x_i < x_i' \Longrightarrow x_i \; P_i \; x_i'.$$

The symmetric and asymmetric parts of R_i are denoted by I_i, P_i, respectively.

For each $i \in I$, let R_i^0 be i's true single-peaked preference. Denote the profile of those true preferences by $R^0 \equiv (R_i^0)_{i \in I}$. The true preference profile R^0 is fixed throughout the analysis and its peak profile is denoted by $p^0 \equiv (p_i^0)_{i \in I}$. We assume that the amount of the available resource is scarce, that is, $\Omega \leq \sum_{i \in I} p_i^0$. However, we can easily deal with the opposite case in a parallel way. An allocation $x \in X$ is *Pareto efficient* if for each $i \in I$, $x_i \leq p_i^0$. [9] Let $P(R^0)$ be the set of *Pareto efficient* allocations.

A *rule* is a function ψ which maps a preference profile R to an allocation $\psi(R) \in X$. In this paper, we restrict our attention to rules that satisfy *peak-only*: for every R, R' such that $p = p'$, $\psi(R) = \psi(R')$. Hence every rule can be seen as a function from $[0, \Omega]^N$ to X, which maps each peak profile $p \in [0, \Omega]^N$ to an allocation $\psi(p) \in X$. Henceforth, we treat a rule ψ as a function that maps a peak profile to an allocation.

When there is no confusion, we do not distinguish a singleton and the member of the singleton. For example, when $A = \{\psi(p)\}$, we may write $A = \psi(p)$.

2.2 Manipulation games

We introduce the direct revelation game of a rule as a tool to analyze the consequence of strategic manipulation –a manipulation game. Given a rule ψ, the direct

[8]The uniform rule is obviously not the only strategy-proof rule in this model once the requirement of efficiency or symmetry is dropped.

[9]Note that this "same-sidedness" definition is equivalent to the usual definition of *Pareto efficiency* which states that no one can gain unless someone loses by any switch of allocations.

revelation game of ψ (henceforth, the *game* of ψ) is the game in which the strategy space of each $i \in I$ is the set of peaks $[0, \Omega]$ and the outcome function is ψ itself. A peak profile $p \in [0, \Omega]^I$ is a *Nash equilibrium* in the game of ψ (according to the true preference profile R^0) if for each $i \in I$ and each $p_i' \in [0, \Omega]$,

$$\psi_i(p) \; R_i^0 \; \psi_i(p_i', p_{-i}).$$

Let $N_e(\psi, R^0)$ be the set of Nash equilibria and let

$$N(\psi, R^0) \equiv \{x \in X : \exists R \in N_e(\psi, R^0), \; x = \psi(R)\}$$

be the set of *Nash equilibrium allocations.* Also, let

$$PN(\psi, R^0) \equiv \{x \in X : \exists p \in N_e(\psi, R^0), \; x = \psi(p) \text{ and } x \in P(R^0)\}$$

be the set of *Pareto efficient Nash equilibrium allocations.* We also consider a stronger notion of Nash equilibrium that is robust to any coalitional deviation. A peak profile $p \in [0, \Omega]^I$ is a *strong Nash equilibrium* in the game of ψ if for each $S \subseteq I$ and each $p_S' \equiv (p_i')_{i \in S} \in [0, \Omega]$, it is not possible that

$$\begin{aligned} \psi_i(p_S', p_{N \setminus S}) \; R_i^0 \; \psi_i(p) \quad \forall i \in S, \\ \psi_j(p_S', p_{N \setminus S}) \; P_j^0 \; \psi_j(p) \quad \exists j \in S. \end{aligned}$$

Let $SN_e(\psi, R^0)$ be the set of strong Nash equilibria and let

$$SN(\psi, R^0) \equiv \{x \in X : \exists R \in SN_e(\psi, R^0), \; x = \psi(R)\}$$

be the set of *strong Nash equilibrium allocations.* [10]

It is easy to see that, if ψ satisfies the very mild condition of "unanimity":

$$\sum_{i \in I} p_i = \Omega \Longrightarrow \psi(p) = p \quad \forall p \in D^N,$$

then

$$SN(\psi, R^0) \subseteq P(R^0),$$

and hence

$$SN(\psi, R^0) \subseteq PN(\psi, R^0) \subseteq N(\psi, R^0)$$

where the last inclusion always holds by definition. However, the non-emptiness of these equilibrium sets are non-trivial.

[10] Although we defined strong Nash equilibrium by weak domination, all our results hold under strong domination. In terms of establishing the existence of equilibrium, this makes the problem difficult, since the set of strong Nash equilibria under weak domination is contained by the set of strong Nash equilibria under strong domination.

2.3 Uniform rule

The rule that has played a prominent role in this problem is the *uniform rule* (Benassy, 1982; Sprumont, 1991):

Uniform rule, U: For each $p \in [0, \Omega]^I$ and each $i \in I$,

$$U_i(p) = \begin{cases} \min\{p_i, \lambda\} \text{ if } \sum_i p_i \geq \Omega, \\ \max\{p_i, \lambda\} \text{ if } \sum_i p_i \leq \Omega, \end{cases}$$

where λ solves $\sum_{j \in I} U_j(p) = \Omega$.

This rule is justified by many desirable properties. [11] Among them, a particularly interesting fact is its robustness to strategic manipulation. [12] Furthermore, under Pareto efficiency and symmetry, this rule is the unique strategy-proof rule (Ching, 1994). [13] In this sense, any other efficient and symmetric rule is strategically manipulable. The following rules are such examples:

Proportional rule, PRO: For each $p \in [0, \Omega]^I$ and each $i \in I$,

$$PRO_i(p) = \begin{cases} \frac{p_i}{\sum_{j \in I} p_j} \cdot \Omega \text{ if } \sum_{j \in I} p_j > 0, \\ \frac{\Omega}{n} \text{ otherwise.} \end{cases}$$

We remark that the proportional rule is discontinuous around the origin, though it is obviously continuous in the interior of the positive orthant: for example, $PRO_1(0,0) = \frac{\Omega}{2}$ and $PRO_1(p_1, 0) = \Omega$ for each $p_1 > 0$. The next variants of the proportional rule are introduced so as to recover continuity:

f-proportional rule, ψ^f: Let $f : [0, \Omega] \to \mathbb{R}_{++}$ be a strictly increasing and continuous function and define the rule ψ^f by,

$$\psi_i^f(p) = \frac{f(p_i)}{\sum_{j \in I} f(p_j)} \Omega \quad \forall p \in [0, \Omega]^I, \ \forall i \in I.$$

Note that the proportional rule is not a f-proportional rule because of the above mentioned discontinuity.

Equal distance rule, ED: for each $p \in [0, \Omega]^I$ and each $i \in I$,

$$ED_i(p) = \begin{cases} \max\{0, p_i - \lambda\} \text{ if } \sum_{j \in I} p_j \geq \Omega, \\ p_i + \frac{\Omega - \sum_{j \in I} p_j}{n} \text{ if } \sum_{j \in I} p_j \leq \Omega, \end{cases}$$

where λ solves $\sum_{j \in I} \max\{0, p_j - \lambda\} = \Omega$.

[11] See Thomson (2005, Ch. 11) for a survey.

[12] See, for example, Sprumont (1991), Ching (1994), and Serizawa (2006).

[13] These three properties are defined as usual. Pareto efficiency and symmetry will be formally introduced later. Strategy-proofness states that no one has an incentive to misreport his preference.

The reason why these rules are manipulable is simple: an agent can profitably increase what he gets by over-reporting his peak. That is, they are manipulable because of their sensitivity to the change of peaks. Notice that the uniform rule does not share this feature.

However, the fact that a rule ψ is manipulable does not imply anything on which allocations are realized through strategic manipulation. Indeed, if we observe that possible manipulations of the rule are not so serious, then we may conclude that its degree of manipulability is small. [14] This motivates us to analyze the consequence of strategic manipulation of rules and its relation to the uniform rule.

3 General equivalence results

Before proceeding to our main results, we need to introduce several straightforward properties that a rule should satisfy. Our main results can be applied to any arbitrary rule satisfying them. Like we said in the previous section, rules are throughout assumed to satisfy "peak-only". We can thus simply define the properties using peaks of preferences.

Pareto efficiency: For each $p \in [0, \Omega]^I$, if $\sum_{i \in I} p_i \geq \Omega$, then for each $i \in I$, $p_i \geq \psi_i(p)$; if $\sum_{i \in I} p_i \leq \Omega$, then for each $i \in I$, $p_i \leq \psi_i(p)$. [15]

Own peak monotonicity: For each $p \in [0, \Omega]^I$, each $i \in I$, and $p'_i \in [0, \Omega]$ such that $p_i \leq p'_i$, we have $\psi_i(p) \leq \psi_i(p'_i, p_{-i})$.

Own peak continuity: For each $i \in I$ and each $p_{-i} \in [0, \Omega]^{I \setminus \{i\}}$, $\psi_i(\ \cdot\ , p_{-i})$ is a continuous function on $[0, \Omega]$.

Others peak monotonicity: For each $p \in [0, \Omega]^I$, each $i, j \in I$, and $p'_i \in [0, \Omega]$ such that $p_i < p'_i$ and $i \neq j$, we have $\psi_j(p'_i, p_{-i}) \leq \psi_j(p)$.

Peak order preservation: For each $p \in [0, \Omega]^I$ and each $i, j \in I$ such that $p_i \leq p_j$, we have $\psi_i(p) \leq \psi_j(p)$.

It is easy to check that all rules defined in the last section and their convex combinations satisfy all of the above five properties. [16] As far as the authors know, all symmetric and continuous rules that have been discussed so far in the literature satisfy these properties. [17]

[14] In models of one indivisible good allocation with monetary transfers, Fujinaka and Sakai (2006a,b) and Sakai (2006) establish such "nearly robustness to manipulation" of manipulable rules.

[15] Note that, by single-peakedness, this "same-sidedness" definition is equivalent to the standard definition saying that no one can gain unless someone loses by switching allocations.

[16] The exception is the proportional rule, since it is not *own peak continuous* at the origin. However, it is *own peak continuous* if agents are restricted to report positive peaks. The next section discusses this point.

[17] There are many different types of rules satisfying all those properties. For example, in the context of the bankruptcy problem where an amount of a divisible resource is to be allocated

Let Ψ^1 be the set of rules satisfying the above five properties. Our first main result shows that for every rule $\psi \in \Psi^1$, the sets of Pareto efficient Nash and strong Nash equilibrium allocations coincide and in fact contain one and only one allocation: it is the uniform allocation. In terms of implementation theory, this result implies that the direct revelation game of any such rule implements the uniform rule in strong Nash equilibria.

Theorem 1. *For every rule ψ satisfying efficiency, own peak monotonicity, others peak monotonicity, peak order preservation, and own peak continuity,*

$$\emptyset \neq SN(\psi, R^0) = PN(\psi, R^0) = U(R^0).$$

Proof. See the Appendix. □

In view of Theorem 1, it is natural to ask whether the set of Nash equilibrium allocations is also the singleton of the uniform allocation. The answer is no in general. For example, Saijo, Sjöström, and Yamato (2003) point out that the direct revelation game of the uniform rule is plagued with "bad" Nash equilibria. To see this, consider the two person case where $p_1^0 < 0.5 < p_2^0$ and $\Omega = 1 < p_1^0 + p_2^0$. Then $p = (0.5, 0.5)$ is a Nash equilibrium but $U(p) = (0.5, 0.5) \notin P(R^0)$. This fact motivates us to investigate under which conditions the same equivalence can be reestablished.

The next condition is a combination of *own weak monotonicity* and its strict version that applies only to situations where an agent is receiving a positive allotment:

Strict own peak monotonicity: ψ is *own peak monotonic* and for each $p \in [0, \Omega]^I$, each $i \in I$, and $p_i' \in [0, \Omega]$ such that $0 < \psi_i(p)$ and $p_i < p_i'$,

$$\psi_i(p) < \psi_i(p_i', p_{-i}).$$

The next theorem shows that, once *own peak monotonicity* in Theorem 1 is strengthened to *strict own peak monotonicity*, such bad Nash equilibria are eliminated and the uniform allocation becomes the unique Nash equilibrium allocation.

Theorem 2. *For every rule ψ satisfying efficiency, strict own peak monotonicity, others peak monotonicity, peak order preservation, and own peak continuity,*

$$\emptyset \neq SN(\psi, R^0) = PN(\psi, R^0) = N(\psi, R^0) = U(R^0).$$

Proof. See the Appendix. □

Let Ψ^2 be the set of rules satisfying the properties of Theorem 2. Obviously, $\Psi^2 \subsetneq \Psi^1$. Every f-proportional rule and the equal loss rule belong to Ψ^2, but the uniform rule does not since it is not *strictly own peak monotonic.* [18] Discussions on our theorems are gathered in the next section.

according to agents' claims, many allocation rules satisfy the counterparts of these properties. Since these allocation rules can be easily translated into rules in our model by regarding the claims as peaks of preferences, rules so obtained in our model also satisfy those properties. An interesting example of such a rule is the Talmud rule (Aumann and Maschler, 1985). For a survey of the bankruptcy problem and its variants, see, Thomson (2003).

[18] For example, when $\Omega = 2$, $U(1, 1) = (1, 1) = U(1, 2)$.

4 Discussions

4.1 Existence of a strong Nash equilibrium

The proof of Theorem 1 involves the existence proof of a strong Nash equilibrium in the direct revelation game of any rule $\psi \in \Psi^1$. In the proof, we first define a "reduced game" where agents outside of the reduced game are fixed to report Ω, and then establish the existence of a Nash equilibrium in the reduced game using a standard fixed-point argument. Then, we show that the Nash equilibrium profile of strategies of agents in the reduced game and the fixed strategies of the outside agents constitute in fact a strong Nash equilibrium of the original game. This technique is quite different from the standard technique in the literature that links the existence of a strong Nash equilibrium with the non-emptiness of the core in a related NTU cooperative game (e.g., Ichiishi, 1993, p39).[19]

4.2 Coalition-proof Nash equilibrium

A coalition-proof Nash equilibrium is a Nash equilibrium that is robust to any "credible" coalitional deviation. [20] By definition, any strong Nash equilibrium is coalition-proof, and any coalition-proof Nash equilibrium is a Nash equilibrium. However, a coalition-proof Nash equilibrium may be Pareto inefficient and a Pareto efficient Nash equilibrium is not coalition-proof in general. Under which conditions these two notions coincide or are related by inclusion is an ongoing topic (e.g., Yi, 1999; Shinohara, 2005). [21] Theorem 2 states that, under a set of conditions including *strict own peak monotonicity*, the set of strong Nash equilibrium allocations coincides with the set of Nash equilibrium allocations. Since the set of coalition-proof Nash equilibrium allocations contains the first set and is contained by the second set, this implies that the three sets are in fact all the same.

The above argument on the equivalence is based on Theorem 2 and hence depends on *strict own peak monotonicity.* In a companion paper, Bochet and Sakai (2006) show that, in the direct revelation game of the uniform rule, the set of coalition-proof Nash equilibrium allocations and the set of Pareto efficient Nash equilibrium allocations coincide with each other, while it itself coincides with the set of strong Nash equilibrum. [22] Since the uniform rule violates *strict own peak monotonicity*, this result suggests that the equivalence may hold even without *strict own peak monotonicity.* However, this question remains open.

[19] Assuming that the number of outcomes is finite and each agent's payoff depends only on the number of agents who choose the same strategy, Konishi, Le Breton, and Weber (1997) establish the existence of a strong Nash equilibrium. Our games satisfy none of these assumptions.

[20] We refer to the seminal work by Bernheim, Peleg, and Whinston (1987) for its precise definition.

[21] Yi and Shinohara study games satisfying strategic substitutability and certain independence conditions. Our games are not such games, and hence we cannot apply their results to the present model.

[22] Saijo, Sjöström, and Yamato (2003) point out that the uniform rule is not "secure" in that it allows Pareto inefficient Nash equilibria. Bochet and Sakai's result implies that the Pareto inefficient equilibria can be eliminated by credible pre-play communication.

4.3 Manipulation of the proportional rule

As already mentioned, the proportional rule is not *own peak continuous*, and hence our results do not cover this rule. However, if we define this rule on $[\varepsilon, \Omega]^I$ for any positive $\varepsilon > 0$, then the proportional rule satisfies all the properties including *own peak continuity*. When each agent is considered to want some positive amount of the resource, this peak restriction is natural since ε can be arbitrary small. Then the same results as Theorems 1 and 2 hold for the proportional rule defined in this way without no essential change in the proofs. Therefore, when the uniform rule is also defined on $[\varepsilon, \Omega]^I$, the game of the proportional rule doubly implements the uniform rule in Nash and strong Nash equilibria. We remark that the compactness of the strategy space $[\varepsilon, \Omega]$ is necessary to establish the existence of a strong Nash equilibrium, and hence we cannot allow for the half-open strategy space $(0, \Omega]$ or $(\varepsilon, \Omega]$.

4.4 Natural implementation

Theorem 2 implies that the direct revelation game of any rule $\psi \in \Psi^2$ doubly implements the uniform rule in Nash and strong Nash equilibria. In particular, the game of each f-proportional rule satisfies many properties attributed to "natural" mechanisms (Dutta, Sen, and Vohra, 1995; Saijo, Tatamitani, and Yamato, 1996) such as continuity of outcome functions (Postlewaite and Wettstein, 1989), compactness of strategy spaces, self-relevancy (Hurwicz, 1960), or the best response property (Jackson, 1992).[23] This suggests that, when the problem of bad Nash equilibria of the uniform rule is serious and pre-play communications to exclude them are not allowed, these rules can be a good tool to realize the uniform allocation. It seems quite interesting to explore this theoretical prediction in laboratory experiments.

4.5 A remark on a misunderstanding

We shall offer a remark for a possible misunderstanding of our results. One may consider that, whenever ψ is Pareto efficient and symmetric, $N(\psi, R^0)$ itself is automatically a Pareto efficient, symmetric, and Maskin monotonic rule, and hence by a result in Dasgupta, Hammond, and Maskin (1979), $N(\psi, R^0)$ is strategy-proof, and so $N(\psi, R^0)$ is the uniform rule by the characterization theorem by Ching (1994). This story contains many errors. First, non-emptiness of $N(\psi, R^0)$ is non-trivial, so it is unclear if $N(\psi, R^0)$ is a rule. Second, single-valuedness of $N(\psi, R^0)$ is non-trivial and in fact can be multi-valued as we observed for the uniform rule, so we cannot apply the theorem by Dasgupta, Hammond, and Maskin. Third, even if ψ is Pareto efficient and symmetric, it is not clear $N(\psi, R^0)$ will inherit the properties satisfied by ψ.

[23] Moreover, direct revelation games are very natural in our context because agents just need to report their peaks and not their entire preference relation. Notice also that these types of mechanisms do not employ unnatural devices common in implementation such as modulo or integer games.

5 Conclusion

Our results reveal how strong the position of the uniform rule is: even if we use the direct revelation mechanism of a manipulable rule, the uniform allocation is realized as a consequence of strategic manipulation. A positive aspect is that, since the uniform allocation is efficient and envy-free, these two properties can still be achieved even under strategic manipulation. A negative aspect is that any distributional objective the uniform allocation violates is never reached. Our results also suggest that simple and natural games can be used to implement the uniform rule. Though the game of the uniform rule itself fails, any manipulable rule in Ψ^2 can be used to doubly implement the uniform rule in Nash and strong Nash equilibrium. In relation to this rather surprising result, we shall close the discussion by mentioning a future work on implementation theory.

As we observed, *strictly own peak monotonicity* is important to Nash implement the uniform rule but the uniform rule itself does not satisfy this property. In this sense the uniform rule is quite insensitive to change in peak announcements. Indeed this insensitivity is the main reason for the uniform rule to satisfy Maskin's monotonicity condition for Nash implementation (Maskin, 1999) and *strategy-proofness*. On the other hand, a *strictly own peak monotonic* rule as an outcome function can Nash implement the uniform rule because of its sensitivity with respect to changes in peaks reported. Although we do not have general relations between a rule to be implemented using a direct mechanism and an implementing outcome function, our finding seems to suggest that there may be certain sensitivity-insensitivity relations between them. We leave this question open for future research.

6 Appendix

6.1 Auxiliary properties

In the proof, we use some properties that do not appear in the statements of the theorems.

Symmetry: for each $p \in [0, \Omega]^I$ and each $i, j \in I$ such that $p_i = p_j$,

$$\psi_i(p) = \psi_j(p).$$

Note that we define this property in a rather strong way, since we require that i and j receive the same allotments, not only indifferent allotments. However, under *efficiency*, the standard definition and this one are equivalent. It is clear that *peak order preservation* implies *symmetry.*

Non-bossiness: for each $p \in [0, \Omega]^I$, each $i \in I$, and $p'_i \in [0, \Omega]$ such that $\psi_i(p'_i, p_{-i}) = \psi_i(p)$,

$$\psi(p'_i, p_{-i}) = \psi(p).$$

This condition is introduced by Satterthwaite and Sonnenschein (1981) and states that no one can change someone else's allotment unless he changes his own. Obviously, *others peak monotonicity* implies *non-bossiness.*

6.2 Proofs of Theorems 1 and 2

Lemma 1. *If ψ satisfies own peak monotonicity, own peak continuity, and non-bossiness, then for each $p \in N_e(\psi, R_0)$, each $i \in I$, and each $p'_i \in [0, \Omega]$ such that $\psi_i(p) < p_i^0$ and $p_i \leq p'_i$,*

$$\psi(p) = \psi(p'_i, p_{-i}).$$

Proof. Let $p \in N_e(\psi, R^0)$, $i \in I$, and $p'_i \in [0, \Omega]$ be such that $\psi_i(p) < p_i^0$ and $p_i \leq p'_i$.

By *own peak monotonicity*, $\psi_i(p) \leq \psi_i(p'_i, p_{-i})$. If $\psi_i(p) < \psi_i(p'_i, p_{-i})$, then by *own peak continuity*, there exists $p''_i \in [0, \Omega]$ such that

$$\psi_i(p) < \psi_i(p''_i, p_{-i}) < \psi_i(p'_i, p_{-i}) \text{ and } \psi_i(p''_i, p_{-i}) \; P_i^0 \; \psi_i(p). \tag{1}$$

However, this contradicts $p \in N_e(\psi, R^0)$. Therefore, $\psi_i(p) = \psi_i(p'_i, p_{-i})$. By *non-bossiness*, $\psi(p) = \psi(p'_i, p_{-i})$. □

Lemma 2. *If ψ satisfies own peak monotonicity, own peak continuity, and non-bossiness, then for each $p \in N_e(\psi, R^0)$, each $i \in I$, and each $p'_i \in [0, \Omega]$ such that $p_i^0 < \psi_i(R)$ and $p'_i \leq p_i$,*

$$\psi(p) = \psi(p'_i, p_{-i}).$$

Proof. This can be shown in a way parallel to the proof of Lemma 1. □

We write Ω_i when i's peak is Ω. This is useful since simply writing Ω does not explain whose peak is Ω.

Lemma 3. *If ψ satisfies own peak monotonicity, own peak continuity, and non-bossiness, then for each $p \in N_e(\psi, R^0)$ with $\psi(p) \in P(R^0)$, whenever,*

$$I_1 \equiv \{i \in I : \psi_i(R) < p_i^0 \text{ and } p_i < \Omega\} \text{ and } I_2 \equiv \{i \in I : \psi_i(R) = p_i^0 \text{ or } p_i = \Omega\},$$

we have

$$\psi(R) = \psi((\Omega_i)_{i \in I_1}, (p_i)_{i \in I_2}) \text{ and } ((\Omega_i)_{i \in I_1}, (p_i)_{i \in I_2}) \in N_e(\psi, R^0).$$

Proof. Let $p \in N(\psi, R^0)$ with $\psi(p) \in P(R^0)$ and let I_1, I_2 be defined as above. Note that $I_1 \cup I_2 = I$ and $I_1 \cap I_2 = \emptyset$.

Step 1: For each $\boldsymbol{i \in I_1}$, $\boldsymbol{\psi(\Omega_i, p_{\ i}) = \psi(p)}$. For each $i \in I_1$, by Lemma 1, $\psi_i(p) = \psi_i(\Omega_i, p_{-i})$, and then by *non-bossiness*, $\psi(p) = \psi(\Omega_i, p_{-i})$.

Step 2: For each $\boldsymbol{i \in I_1}$, $\boldsymbol{(\Omega_i, p_{\ i}) \in N_e(R^0)}$. Let $i \in I_1$. Let us verify that $(\Omega_i, p_{-i}) \in N_e(R^0)$. Since $\psi_i(\Omega_i, p_{-i}) = \psi_i(p) < p_i^0$, for i to profitably deviate at $\psi_i(\Omega_i, p_{-i})$, i needs to increase what he gets. However, since i is already reporting Ω, this is impossible by *own peak monotonicity.* Clearly, every $j \in I_2$ with $\psi_j(R) = p_j^0$ has no incentive to deviate. Also, no $j \in I_2$ with $p_j = \Omega$ can profitably deviate at (Ω_i, p_{-i}), since $p_j = \Omega$ and ψ is *own peak monotonic.*

It remains to show that no $j \in I_1$ with $j \neq i$ has an incentive to deviate at (Ω_i, p_{-i}). Suppose, by contradiction, that there exists $j \in I_1$ with $j \neq i$ such that for some $p'_j \in [0, \Omega]$,

$$\psi_j(\Omega_i, p_j, p_{-ij}) < \psi_j(\Omega_i, p'_j, p_{-ij}). \tag{2}$$

By *own peak monotonicity*, $p_j < p'_j$. By Step 1,

$$\psi_j(p_i, \Omega_j, p_{-ij}) = \psi_j(p_i, p_j, p_{-ij}) = \psi_j(\Omega_i, p_j, p_{-ij}). \tag{3}$$

By *others peak monotonicity*,

$$\psi_j(\Omega_i, p'_j, p_{-ij}) \leq \psi_j(p_i, p'_j, p_{-ij}). \tag{4}$$

By (2), (3), and (4),

$$\psi_j(p_i, \Omega_j, p_{-ij}) < \psi_j(p_i, p'_j, p_{-ij}), \tag{5}$$

which contradicts *own peak monotonicity.*

Step 3: Concluding. By inductively applying Steps 1 and 2, we obtain

$$\psi(R) = \psi((\Omega_i)_{i\in I_1}, (p_i)_{i\in I_2}) \text{ and } ((\Omega_i)_{i\in I_1}, (p_i)_{i\in I_2}) \in N_e(\psi, R^0). \tag{6}$$

□

Lemma 4. *If ψ satisfies efficiency, own peak monotonicity, own peak continuity, peak order preservation, and non-bossiness, then for each $p \in N_e(\psi, R^0)$ with $\psi(p) \in P(R^0)$,*

$$\psi(p) = U(R^0).$$

Proof. Let $p \in N_e(\psi, R^0)$ with $\psi(p) \in P(R^0)$. Let

$$\begin{aligned} I_1 &\equiv \{i \in I : \psi_i(R) < p_i^0 \text{ and } p_i < \Omega\}, \\ I_2 &\equiv \{i \in I : \psi_i(R) = p_i^0\}, \\ I_3 &\equiv \{i \in I : p_i = \Omega\}. \end{aligned}$$

Note that $I_1 \cup I_2 \cup I_3 = I$ and they are mutually disjoint. By Lemma 3,

$$\begin{aligned} \psi(R) = \psi((\Omega_i)_{i\in I_1}, (p_i)_{i\in I_2}, (\Omega_i)_{i\in I_3}), \\ ((\Omega_i)_{i\in I_1}, (p_i)_{i\in I_2}, (\Omega_i)_{i\in I_3}) \in N_e(\psi, R^0). \end{aligned}$$

By *peak order preservation* to $\psi(R) = \psi((\Omega_i)_{i\in I_1}, (p_i)_{i\in I_2}, (\Omega_i)_{i\in I_3})$, we have,

$$p_k^0 = x_k \leq x_i = x_j < \min\{p_i^0, p_j^0\} \quad \forall i, j \in I_1 \cup I_3, \forall k \in I_2,$$

where the last inequality follows from the *Pareto efficiency* of the Nash equilibrium outcome $\psi(p)$. This immediately implies that

$$x_i = \{p_i^0, \lambda\} \quad \forall i \in I, \tag{7}$$

where $\lambda = x_j$ with $j \in I_1 \cup I_3$. Hence $x = U(R^0)$. Since $x = \psi(p)$ by its definition, $\psi(p) = U(R^0)$. □

Lemma 4 shows that, if $P(R^0) \neq \emptyset$, then any Pareto efficient Nash equilibrium allocation is the uniform allocation. To establish the existence, we in fact prove a stronger statement: there exists a strong Nash equilibrium that supports the uniform allocation.

Lemma 5. *If ψ satisfies efficiency, own peak monotonicity, others peak monotonicity, own peak continuity, and peak order preservation, then there exists $p \in SN_e(\psi, R^0)$ such that $\psi(p) = U(R^0)$.*

Proof. **Step 1: Setting up.** Let $z \equiv U(R^0)$,

$$S \equiv \{i \in I : z_i = p_i^0\} \text{ and } T \equiv \{i \in I : z_i < p_i^0\}.$$

Note that $S \cup T = I$, $S \cap T = \emptyset$, $\max_{i\in S} p_i^0 < \min_{i\in T} p_i^0$, and

$$\max_{i\in S} z_i < \min_{i\in T} z_i. \tag{8}$$

Step 2: Finding a Nash equilibrium in a reduced game. We shall consider a reduced game of the members of S given that every $i \in T$ reports Ω. For each $i \in S$, let $u_i : [0, \Omega] \to \mathbb{R}$ be a continuous representation of R_i^0; also, for each $(p_j)_{j\in S} \in [0, \Omega]^S$, define i's "payoff function" $v_i : [0, \Omega]^S \to [0, \Omega]$ by,

$$v_i(p) = u_i(\psi_i((p_j)_{j\in S}, (\Omega_j)_{j\in T})). \tag{9}$$

By *own peak continuity* of ψ and continuity of u_i, v_i is continuous.

Given

$$(p_j)_{j\in S\setminus\{i\}} \in [0, \Omega]^{S\setminus\{i\}}, \tag{10}$$

for every $p_i', p_i'' \in [0, \Omega]$ with $p_i' < p_i''$, by *own peak monotonicity* of ψ,

$$\begin{aligned} \psi_i(p_i'', (p_j)_{j\in S\setminus\{i\}}, (\Omega_j)_{j\in T}) \geq \\ \psi_i(\alpha p_i' + (1-\alpha)p_i'', (p_j)_{j\in S\setminus\{i\}}, (\Omega_j)_{j\in T}) \geq \\ \psi_i(p_i', (p_j)_{j\in S\setminus\{i\}}, (\Omega_j)_{j\in T}), \end{aligned}$$

and by single-peakedness of u_i,

$$\begin{aligned} u_i(\psi_i(\alpha p_i' + (1-\alpha)p_i'', (p_j)_{j\in S\setminus\{i\}}, (\Omega_j)_{j\in T}) \geq \\ \min\{u_i(\psi_i(p_i', (p_j)_{j\in S\setminus\{i\}}, (\Omega_j)_{j\in T})), u_i(\psi_i(p_i'', (p_j)_{j\in S\setminus\{i\}}, (\Omega_j)_{j\in T}))\}. \end{aligned}$$

Thus v_i is quasi-concave in $p_i \in [0, \Omega]$.

Consider the game in which the set of players is S, the strategy space of each $i \in S$ is $[0, \Omega]$, and the payoff function of each $i \in S$ is v_i. Since $[0, \Omega]$ is compact and convex and v_i is continuous in $[0, \Omega]^S$ and quasi-concave in $[0, \Omega]$, by a standard fixed-point argument, there exists a Nash equilibrium $(p_i)_{i\in S} \in [0, \Omega]^S$ in this game.

Step 3. Characterizing the Nash equilibrium allocation. Let $x \equiv \psi((p_j)_{j\in S}, (\Omega_j)_{j\in T})$. Let $\alpha \equiv x_k$ for $k \in T$. The assumption $\Omega < \sum_{j\in I} p_j^0$ implies $T \neq \emptyset$, and hence by *efficiency*,

$$\psi_i(0, (p_j)_{j\in S\setminus\{i\}}, (\Omega_j)_{j\in T}) = 0 \;\; \forall i \in S. \tag{11}$$

We first claim that for each $i \in S$, $x_i = p_i^0$. If there is $i \in S$ with $p_i^0 < x_i$, then (11) and *own peak continuity* imply that i could decrease what he gets by reporting some $p_i' \in (0, p_i)$, a contradiction. Hence for each $i \in S$, $x_i \leq p_i^0$. Let $S_1 \equiv \{i \in S : x_i = p_i^0\}$ and $S_2 \equiv \{i \in S : x_i < p_i^0\}$. We shall show that $S_2 = \emptyset$ by a contradiction argument. Suppose not, $S_2 \neq \emptyset$. For each $i \in S_2$, by *own peak monotonicity* and the fact that p_S is a Nash equilibrium, we have $x_i = \psi_i(\Omega_i, p_{S\setminus\{i\}}, \Omega_T)$, and by *non-bossiness*, $x = \psi(\Omega_i, p_{S\setminus\{i\}}, \Omega_T)$, and then by *equal treatment of equals*, $x_i = \alpha$. Therefore

$$x_i = p_i^0 = z_i \;\; \forall i \in S_1, \tag{12}$$
$$\alpha = x_i < p_i^0 = z_i \;\; \forall i \in S_2, \tag{13}$$
$$\alpha = x_i < p_i^0 \;\; \forall i \in T. \tag{14}$$

Since now there exists some $j \in S_2$, by feasibility, there exists $k \in T$ such that $z_k < x_k$. But, then $z_k < x_k = \alpha = x_j < z_j$, a contradiction to (8). Thus we have shown that for each $i \in S$, $x_i = p_i^0$.

By *symmetry* to $\psi((p_j)_{j\in S}, (\Omega_j)_{j\in T})$,

$$\psi_i((p_j)_{j\in S}, (\Omega_j)_{j\in T}) = \frac{\Omega - \sum_{j\in S} p_i^0}{|T|} \;\; \forall i \in T. \tag{15}$$

and by *peak order preservation* to $\psi((p_j)_{j\in S}, (\Omega_j)_{j\in T})$,

$$\max_{j\in S} p_j < \frac{\Omega - \sum_{j\in S} p_i^0}{|T|}. \tag{16}$$

Overall, by (15) and (16),

$$\psi_i((p_j)_{j\in S}, (\Omega_j)_{j\in T}) = \min\{p_i^0, \frac{\Omega - \sum_{j\in S} p_i^0}{|T|}\} \;\; \forall i \in I. \tag{17}$$

Therefore $\psi((p_j)_{j\in S}, (\Omega_j)_{j\in T}) = U(R^0)$.

Step 4. Concluding. Let $p \equiv ((p_j)_{j\in S}, (\Omega_j)_{j\in T})$. It remains to show that, in the direct revelation game of ψ, p is a strong Nash equilibrium.

Suppose, by contradiction, that there exist $I' \subseteq I$ and $p'_{I'} \equiv (p_i')_{i\in I'} \in [0, \Omega]^{I'}$ such that

$$\psi_i(p'_{I'}, p_{I\setminus\{I'\}}) \; R_i \; \psi_i(U_i(p)) \;\; \forall i \in I', \tag{18}$$
$$\psi_i(p'_{I'}, p_{I\setminus\{I'\}}) \; P_j \; \psi_i(p) \;\; \exists j \in I', \tag{19}$$
$$p_i' \neq p_i \;\; \forall i \in I'. \tag{20}$$

Since $\psi(p) = U(R^0)$,

$$\psi_i(p'_{I'}, p_{I\setminus\{I'\}}) \geq \psi_i(U_i(p)) \ \ \forall i \in I', \tag{21}$$
$$\psi_j(p'_{I'}, p_{I\setminus\{I'\}}) > \psi_j(p) \ \ \exists j \in I'. \tag{22}$$

Obviously, $\psi_j(p) < \psi_j(p'_{I'}, p_{I\setminus I'})$, and so $p_j = \Omega$. Let

$$A \equiv \{i \in I' : p_i < p'_i\}, \tag{23}$$
$$B \equiv \{i \in I' : p'_i < p_i\}. \tag{24}$$

Note that $A \cup B = I'$ and $A \cap B = \emptyset$. Note also that, by $p_j = \Omega$,

$$j \in B. \tag{25}$$

By repeatedly applying *others peak monotonicity*,

$$\psi_i(p) \leq \psi_i(p_A, p'_B, p_{I\setminus I'}) \ \ \forall i \in I \setminus B, \tag{26}$$
$$\psi_i(p'_A, p'_B, p_{I\setminus I'}) \leq \psi_i(p_A, p'_B, p_{I\setminus I'}) \ \ \forall i \in I \setminus A. \tag{27}$$

For each $i \in B$, if

$$\psi_i(p_A, p'_B, p_{I\setminus I'}) < \psi_i(p), \tag{28}$$

then by (27),

$$\psi_i(p'_A, p'_B, p_{I\setminus I'}) < \psi_i(p), \tag{29}$$

which contradicts (18). Hence

$$\psi_i(p) \leq \psi_i(p_A, p'_B, p_{I\setminus I'}) \ \ \forall i \in B, \tag{30}$$

and so by (26),

$$\psi_i(p) = \psi_i(p_A, p'_B, p_{I\setminus I'}) \ \ \forall i \in I. \tag{31}$$

By (27) and (31),

$$\psi_i(p'_A, p'_B, p_{I\setminus I'}) \leq \psi_i(p) \ \ \forall i \in B. \tag{32}$$

However, this contradicts (22) and (25). □

Proof of Theorem 1. By Lemma 4, $PN(\psi, R^0) \subseteq U(R^0)$. By Lemma 5, $\emptyset \neq U(R^0) \subseteq SN(\psi, R^0)$. Since $SN(\psi, R^0) \subseteq PN(\psi, R^0)$ by *Pareto efficiency*,

$$\emptyset \neq SN(\psi, R^0) = PN(\psi, R^0) = U(R^0). \tag{33}$$

□

Lemma 6. *If ψ satisfies efficiency, strict own peak monotonicity, others peak monotonicity, own peak continuity, and peak order preservation, then*

$$PN(\psi, R^0) = N(\psi, R^0).$$

Proof. It suffices to show that for each $p \in N_e(\psi, R^0)$ and each $i \in I$, $\psi_i(p) \leq p_i^0$. Suppose, on the contrary, that there exists $j \in I$ such that $p_j^0 < \psi_j(p)$. By $\Omega \leq \sum_{i \in I} p_i^0$, there exists $k \in I$ such that $\psi_k(p) < p_k^0$. Note that $0 < \psi_j(p)$ and $0 < p_k^0$. If $0 < p_j$, then *strict own peak monotonicity* and *own peak continuity* together imply that there exists $p_j' \in (0, p_j)$ such that

$$\psi_j(p_j', p_{-j}) \ P_j^0 \ \psi_j(p), \tag{34}$$

a contradiction to $p \in N_e(\psi, R^0)$. Hence $p_j = 0 < \psi_j(p)$. This and *efficiency* of ψ together imply $p_k \leq \psi_k(p)$. Summarizing,

$$0 \leq p_k \leq \psi_k(p) < p_k^0 \leq \Omega. \tag{35}$$

If $0 < \psi_k(p)$, then *strict own peak monotonicity* and *own peak continuity* together imply that k could increase what he receives by announcing some $p_k' \in (p_k, \Omega]$, a contradiction. Therefore

$$0 = p_k = \psi_k(p). \tag{36}$$

Note that *symmetry* and *others peak monotonicity* together imply

$$\frac{\Omega}{n} \leq \psi_k(\Omega_k, p_{-k}) \geq . \tag{37}$$

By (35), (36), and (37),

$$\psi_k(p) = 0 < \frac{\Omega}{n} \leq \psi_k(\Omega_k, p_{-k}) \text{ and } p_k = 0 < p_k^0. \tag{38}$$

Therefore by *own peak continuity*, there exists $p_k' \in (0, \Omega)$ such that

$$\psi_k(p_k', p_{-k}) \ P_k^0 \ \psi_k(p), \tag{39}$$

a contradiction to $p \in N_e(\psi, R^0)$. □

Proof of Theorem 2. Implied by Theorem 1 and Lemma 6. □

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