Equal-Budget Choice Equivalent Solutions in Exchange Economies[*]

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Abstract

Given a family of linear budget sets, an allocation is *equal opportunity equivalent* (Thomson, 1994) if there exists a common budget set such that each agent is indifferent between the bundle that he gets and the best bundle he can obtain in the choice set. We first study the robustness properties of *equal opportunity equivalent* correspondences with respect to change in preferences. We impose *independence to irrelevant preference changes* and connect this property with the *implementation of rules via some game-theoretic solution concept*. We provide an equivalence result with the equal-income Walrasian rule. Next, we study robustness with respect to change in the number of agents and derive a characterization of the equal-income Walrasian rule. Our results provide additional justifications for the equal-division of resources as a first step toward fairness.

Keywords: Fair allocations, Equal-income Walrasian solution, Equal-budget choice equivalence, Robustness, Implementability via some game theoretical solution concept.

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1 Introduction

The literature on *fair allocations* is concerned with the division of several goods among agents with identical claims on the goods but different preferences. The model we use is that of exchange economies with a variable number of agents and infinitely divisible goods. An allocation rule that is central in this context is the *equal-income Walrasian rule.* While there are obviously different notions of fairness and equity (e.g. *envy-freeness*, *egalitarian equivalence*), our position will be that of *equality of choice* (Thomson, 1994). Given a *family of choice sets* $\mathcal{B}$, we say that an allocation is *fair* if there exists a common choice set in that family such that each agent is indifferent between the bundle he is assigned and the best bundle he can obtain in the so-called choice set.[1] This fairness property is termed *equal-opportunity equivalence with respect to* $\mathcal{B}$ (Thomson, 1994).[2] Interestingly, by appropriately choosing the family of choice sets, this notion of fairness generalizes many concepts used in the literature on fair allocations. For instance, if $\mathcal{B} = \left\{x_0 : x_0 \in \mathbb{R}^{\ell}_{+}\right\}$, then the set of allocations that are equal-opportunity equivalent with respect to $\mathcal{B}$ coincides with the set of *egalitarian equivalent* allocations, a fairness concept introduced by Pazner and Schmeidler (1978). On the other hand, if each choice set is now an allocation instead of a bundle, i.e. if $\mathcal{B} = \left\{x : x \in \mathbb{R}^{\ell n}_{+}\right\}$, the set of allocations that are equal-opportunity equivalent with respect to $\mathcal{B}$ coincides with the set of *envy-free* allocations, a fairness concept introduced by Foley (1968).

While families of choice sets are obviously numerous, we will restrict our attention to a family that plays an important role in economics, namely the *family of linear choice sets.* Given a common reference bundle α, a family of linear choice sets $\mathcal{B}_\alpha$ is then parametrized by price vectors. An allocation rule is *equal-budget choice equivalent* with respect to a family of linear choice sets $\mathcal{B}_\alpha$ if, for each preference profile, and for each allocation x that the rule selects, there exists a price vector and a choice set such that each agent is indifferent between the bundle he gets at x and the best bundle he can afford in the common choice set.

Equal-budget choice equivalence combines idea of *egalitarian-equivalence*

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[1] See also Nicolò and Perea (2005).

[2] In addition to the notion of *equal-opportunity equivalence*, Thomson (1994) first introduces the notion of *equal-opportunity.* The former obviously gives allocation rules that, in general, selects more allocations than the latter. Moreover, the former combines the idea of *equal-opportunities* with that of *egalitarian-equivalence.*

with that of *equal-opportunity* and *Walrasian equilibrium*. To fix ideas about the *equal-budget choice equivalent* allocation rules generated by such families, let α be the point of equal division of the social endowment. Then, the *equal-budget choice equivalent* allocation rule generated is the *equal-income Walrasian rule*.[3] It has often been advocated that the point of equal division of resources is a first step towards equity and fairness. We wish to provide additional support to this claim.

We complement fairness with three robustness requirements, the first one with respect to changes in preferences and the other two with respect to changes in the number of agents, when the feasible set is modified in an appropriate way. The former requirement is called *independence to irrelevant preference changes*. It states the invariance of an allocation rules with respect to change in preferences occurring only outside of the feasible set. Interestingly, this property has a clear link with the decentralization of allocation rules –i.e. their full implementation. After all, once allocation rules satisfying desirable properties have been identified, the objective of a planner is to check whether there exist institutions–non-cooperative games–through which these allocation rules emerge as the equilibrium set of such games.

In fact, *independence to irrelevant preference changes* is a necessary condition for the *implementation of a rule via some game-theoretic solution concept*. We emphasize this connection and show that when preferences are continuous, monotonic and convex, then no allocation rule satisfies this requirement except when α belongs to the set of corners of the Edgeworth box. Under this preference domain, the *equal-income Walrasian rule* is not implementable via any solution concept. However, by adding differentiability, it becomes implementable in subgame perfect equilibrium. We show that the only rule that satisfies *equal-budget choice equivalence*, *independence to irrelevant change in preferences* and *neutrality* is the *equal-income Walrasian rule*.

Next, we turn to robustness with respect to a change in the number of agents. The second robustness property we study is termed *consistency* (see Thomson, 1988). We show that the only rule that satisfies *equal-budget choice equivalence* and *consistency* is the *equal-income Walrasian rule*.

Finally, our last property is called *replication invariance* (see Thomson, 1988). We show that a *Pareto efficient* selection of an *equal-budget choice equivalent* rule satisfies *consistency* and *replication invariance* if and only if

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[3] See Proposition 1 in section 3 for a formal proof of this statement.

it is a subrule of the *equal-income Walrasian rule.* This result is in line with Theorem 3 in Thomson (1988).

The paper proceeds as follows. In Section 2, we introduce the necessary definitions and notations. In Section 3, we present the main results and offer some concluding remarks in Section 4.

2 Definitions and notations

There is a set $N \equiv \{1, ..., n\}$, $n \geq 2$, of *agents* drawn from an infinite population $\mathcal{N}$, and a set $L = \{1, ..., \ell\}$ of *infinitely divisible goods.* For each $i \in N$, $\mathbb{R}^\ell_+$ is agent i's *consumption set.* For each $i \in N$, let R_i be a complete and transitive binary relation on $\mathbb{R}^\ell_+$ indicating (weak) *preferences.* The associated *strict preference* and *indifference relations* are denoted by P_i and I_i. For each $i \in N$, let $\mathcal{R}_i$ be i's set of possible preferences. Let $\mathcal{R}^N = \prod_{i \in N} \mathcal{R}_i$ be the set of possible *preference profiles.* A typical preference profile is a list $R = (R_i)_{i \in N} \in \mathcal{R}^N$. For each $R \in \mathcal{R}^N$ and each $M \subset N$, let $R_M = (R_i)_{i \in M}$ be the profile of preferences restricted to $M \subset N$.

Let $\bar{\omega} \gg 0$ be the *social endowment.* Once the set N of agents is fixed, $\bar{\omega}$ is also fixed across preference profiles. An economy is thus simply a list of preference relations and an aggregate endowment. Formally an economy is a list $(R, \bar{\omega}) \in E^N = \{\mathcal{R}^N \times \mathbb{R}^\ell_+ \colon n \in \mathbb{N}\}$. We consider two domains of economies E^N_C and E^N_D.[4]

Class of preferences $\mathcal{R}^N_C$: For each agent $i \in N$, each $R_i \in \mathcal{R}_i$ is continuous, convex and monotonic.[5]

Class of preferences $\mathcal{R}^N_D$: For each agent $i \in N$, each $R_i \in \mathcal{R}_i$ is convex, monotonic and representable by a differentiable utility function.

Given $\bar{\omega} \gg 0$, a *(feasible) allocation* is a list of *bundle* $(x_i)_{i \in N} \in \mathbb{R}^{\ell n}_+$ such that $\sum x_i \leq \bar{\omega}$. Given $i \in N$ and $x_i \in \mathbb{R}^\ell_+$, let $x^l_i \in \mathbb{R}_+$ be the *lth coordinate* of x_i. For each $x \in \mathbb{R}^{\ell n}_+$ and each $M \subset N$, let x_M be the sub-allocation

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[4] R^N_C is the domain of *classical preferences*, while for R^N_D is the *subdomain of classical preferences* that are representable by a *differentiable utility function.*

[5] A preference relation R_i defined over $\mathbb{R}^L_+$ is convex if, for every x_i and $y_i \in \mathbb{R}^L_+$ such that $x_i \; P_i \; y_i$, we have that $\lambda x_i + (1-\lambda) y_i \; P_i \; y_i$ for every $\lambda \in (0, 1]$.

A preference relation R_i defined over $\mathbb{R}^L_+$ is monotonic if, for each x_i and $y_i \in \mathbb{R}^L_+$, $x_i \gg y_i$ implies that $x_i \; P_i \; y_i$.

restricted to agents in M. Likewise, for each $x \in \mathbb{R}_{+}^{\ell n}$ and each $M \subset N$, let $x_{N \setminus M}$ be the sub-allocation restricted to agents in $N \setminus M$.

Fix $\bar{\omega} \gg 0$. Let $A(\bar{\omega}) \equiv \{x \in \mathbb{R}_{+}^{\ell n} : \sum x_i \leq \bar{\omega}\}$ be the *set of feasible allocations.*

For each $i \in N$, let $A_i(\bar{\omega})$ be the projection of $A(\bar{\omega})$ onto agent i's consumption set.

A bundle $x_i \in A_i(\bar{\omega})$ is an *orthant-corner* of $A_i(\bar{\omega})$ if $x_i \in \partial\mathbb{R}_{+}^{\ell} \cap A_i(\bar{\omega})$ and there exists $l \in L$ such that $x_i^l = \bar{\omega}^l$ –which implies that for each $k \neq l$, $x_i^k = 0$. We say that x_i is a *trivial-corner* of $A_i(\bar{\omega})$ if for each $l = 1, ..., \ell$ either $x_i^l = \bar{\omega}^l$ or $x_i^l = 0$.

An allocation $x \in A(\bar{\omega})$ is an orthant-corner of $A(\bar{\omega})$ if there exists $i \in N$ such that x_i is an orthant-corner $A_i(\bar{\omega})$. Trivial corners of $A(\bar{\omega})$ are defined in the same fashion. For each $\bar{\omega} \gg 0$, let $C(A(\bar{\omega}))$ be the *set of orthant-corners* $A(\bar{\omega})$. Abusing language, we say that a bundle x_i is an element of $C(A(\bar{\omega}))$ if it is an orthant-corner of $A_i(\bar{\omega})$. From now on, we refer to orthant-corners simply as corners. This should cause no confusion.

Let $\Delta \equiv \{p \in \mathbb{R}_{+}^{\ell} : \sum p_l = 1\}$ be the *price simplex.* For each $i \in N$, each $\alpha \in \mathbb{R}_{+}^{\ell}$, and each $p \in \Delta$, let $B_i(p, \alpha) \equiv \{x_i \in A_i(\bar{\omega}) : p \cdot x_i \leq p \cdot \alpha\}$ be i's *budget set* at price p and bundle α.

For each $i \in N$, each $R_i \in \mathcal{R}_i$, and each $x_i \in \mathbb{R}_{+}^{\ell}$, let $LC(R_i, x_i) \equiv \left\{y_i \in \mathbb{R}_{+}^{\ell} : x_i \; R_i \; y_i\right\}$ be the *lower contour set* of R_i at x_i; $UC(R_i, x_i) \equiv \left\{y_i \in \mathbb{R}_{+}^{\ell} : y_i \; R_i \; x_i\right\}$ be the *upper contour set* of R_i at x_i; $SLC(R_i, x_i) \equiv \left\{y_i \in \mathbb{R}_{+}^{\ell} : x_i \; P_i \; y_i\right\}$ be the *strict lower contour set* of R_i at x_i, and $I(R_i, x_i) \equiv \left\{y_i \in \mathbb{R}_{+}^{\ell} : x_i \; I_i \; y_i\right\}$ be the *indifference curve* of R_i through x_i.

We are interested in *allocation rules* satisfying several desirable properties. An allocation rule $f : E^N \times \bar{\omega} \twoheadrightarrow A(\bar{\omega})$ is a correspondence that associates to each preference profile a set of allocations. For each $i \in N$, each $R_i \in \mathcal{R}_i$, each $\alpha \in \mathbb{R}_{+}^{\ell}$, and each $p \in \Delta$, let $\arg\max R_i|_{B_i(p,\alpha)}$ be the set of *maximizers* of R_i over $B_i(p, \alpha)$.

Definition: $x \in A(\bar{\omega})$ is an *equal-income Walrasian allocation* for $(R, \bar{\omega}) \in E^N$ if there exists $p \in \Delta$ such that for each $i \in N$, $x_i \in B_i(p, \omega_i)$ and $x_i \in \arg\max R_i|_{B_i(p, \frac{\bar{\omega}}{n})}$.

The *equal-income Walrasian correspondence* $W_E : E^N \twoheadrightarrow A(\bar{\omega})$ associates to each economy $(R, \bar{\omega})$ its set of *equal-income Walrasian allocations.*

Definition: For each $(R, \bar{\omega}) \in E^N$, $x \in A(\bar{\omega})$ is *(Pareto) efficient* if there does not exist $y \in A(\bar{\omega})$ such that for each $i \in N$, $y_i \; R_i \; x_i$, and for some $j \in N$, $y_j \; P_j \; x_j$.

For each $(R, \bar{\omega}) \in E^N$, we say that $x \in A(\bar{\omega})$ is derived from a *choice set* $B \subset \mathbb{R}_+^L$ if for each $i \in N$, we have that $x_i \in B$. Let $\mathcal{B}$ be a family of choice sets.

Definition: For each $(R, \bar{\omega}) \in E^N$, $x \in A(\bar{\omega})$ is *equal-opportunity equivalent relative to the family* $\mathcal{B}$ *for* $(R, \bar{\omega})$ if there exists $B \in \mathcal{B}$ such that for each $i \in N$ and for each $x_i^* \in \arg\max R_i|_B$, $x_i \; I_i \; x_i^*$.

The definition of a family of choice sets we gave is very broad. We will restrict our attention to families of *linear choice sets*.

A family of linear choice sets is indexed by a bundle $\alpha \equiv \alpha_{\bar{\omega}} \cdot \bar{\omega} = (\alpha_{\bar{\omega}_1} \cdot \bar{\omega}_1, ..., \alpha_{\bar{\omega}_\ell} \cdot \bar{\omega}_\ell)$, with, for each $l \in L$, $\alpha_{\bar{\omega}_l} \in [0, 1]$, and parametrized by price vectors $p \in \Delta$.[6] Once a point $\alpha \in \mathbb{R}_+^\ell$ has been fixed, the family of choice sets contains each budget set generated by each $p \in \Delta$ and α. That is, given $\alpha \in \mathbb{R}_+^\ell$ and $p \in \Delta$, let $B_\alpha(p) \equiv \left\{z \in \mathbb{R}_+^\ell : p \cdot z \leq p \cdot \alpha\right\}$ be a choice set and $\mathcal{B}_\alpha \equiv \{B_\alpha(p) : p \in \Delta\}$ be the family of α-linear choice sets.

We are interested in several desirable properties of allocation rules. Our first requirement is an axiom of fairness.

Equal-budget choice equivalence: A solution f satisfies *equal-budget choice equivalence relative to* $\mathcal{B}_\alpha$ if,

1) For each $(R, \bar{\omega}) \in E^N$, each $x \in f(R, \bar{\omega})$ is *equal-opportunity equivalent* relative to the family $\mathcal{B}_\alpha$ for $(R, \bar{\omega})$.

2) There does not exist an *equal-opportunity equivalent allocation* y *relative to* $\mathcal{B}_\alpha$ for $(R, \bar{\omega})$ such that $y \notin f(R, \bar{\omega})$.

Given α, let f_α be the *equal-budget choice equivalent* allocation rule relative to $\mathcal{B}_\alpha$. Our next property requires a form of robustness with respect to change in preferences. It states the invariance of a rule to changes in preferences occurring only outside of the feasible set of allocations.

Independence to irrelevant preference changes: A solution f satisfies *independence to irrelevant preference changes* if for each $\{(R, \bar{\omega}), (R', \bar{\omega})\} \subset$

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[6] The reference point can be written as α or $\alpha_{\bar{\omega}} \cdot \bar{\omega}$. For convenience, we use both notations in the paper but it should not cause confusion.

E^N and each $x \in f(R, \bar{\omega})$,

$$[R \neq R' \text{ and } R_i \cap A_i(\bar{\omega}) = R'_i \cap A_i(\bar{\omega}) \text{ for each } i \in N] \implies [x \in f(R', \bar{\omega})].$$

Our third property states the robustness of a rule with respect to changes in the number of agents when the feasible set is modified in an appropriate way. We define the property directly with respect to rules f_α : when a group of agents leaves the economy, not only they modify the feasible set of allocations, but they also modify the initial reference point α.

Consistency: An allocation rule f_α satisfies *consistency* if for each $(R, \bar{\omega}) \in E^N$, each $x \in f(R, \bar{\omega})$ and each $M \subset N$,

$$x_{N \setminus M} \in f_{\alpha_{\bar{\omega}} \cdot x_{N \setminus M}}(R \setminus R_M, \sum_{i \in N \setminus M} x_i).$$

Consistency pertains to a decreasing number of agents. An allocation rule is *consistent* if and only if any sub-allocation of a selected allocation is also selected in the "reduced" economy defined by the corresponding subset of agents and the set of allocations which can be considered as feasible for this subset of agents. An allocation is feasible for a subset of agents if it is possible to reach it while guaranteeing the selected consumption plans to each agent outside the considered subset of agents. Notice in the definition how the initial point α is also "scaled-down". Recall that $\alpha = (\alpha_{\bar{\omega}_1} \cdot \bar{\omega}_1, ..., \alpha_{\bar{\omega}_L} \cdot \bar{\omega}_L)$. The point $\alpha_{\bar{\omega}} \cdot \bar{\omega}_{N \setminus M}$ is,

$$\alpha_{\bar{\omega}} \cdot x_{N \setminus M} = \big(\alpha_{\bar{\omega}_1} \cdot (\bar{\omega}_1 - x^1_M), ..., \alpha_{\bar{\omega}_L} \cdot (\bar{\omega}_L - x^\ell_M)\big).$$

We complement *consistency* with a robustness axiom that applies to situations in which the number of agents increases, and when the feasible set is modified in an appropriate way. Thomson (1988) defines four robustness properties of rules with respect to an increase in the number agents. Among these, the one that has been the object of most of the attention in the literature is *replication invariance* that we define next.

Replication invariance: An allocation rule f_α satisfies *replication invariance* if for each $(R, \bar{\omega}) \in E^N$, each $x \in f(R, \bar{\omega})$, and each $\lambda \in \mathbb{N}$,

$$\lambda x \in f_{\alpha_{\bar{\omega}} \cdot \lambda \bar{\omega}}(\lambda R, \lambda \bar{\omega}).$$

Apart from fairness and robustness, a central authority–e.g. a planner–is interested in the possibility of obtaining the allocations selected by the rule as equilibrium outcomes of some non-cooperative game–i.e. through an institution. Our point of view here is that decentralization entails full implementation of a solution. Before introducing the property of *implementability via some game-theoretic solution concept*, let us add some necessary definitions.

Let T be a *game tree*. Let S be the set of *nodes* of T, s_0 the *initial node*, and Z the set of *terminal nodes*. For each $i \in N$, let M_i be the set of *(pure) strategies*, and for each $s \in S$, let M_i^s be the set of strategies available to i at node $s \in S$. Let $M \equiv \prod_i M_i$ be the set of *strategy profiles*. As is common in the implementation literature, we confine our attention to pure strategies. Let g, the *outcome function*, be a function that associates a feasible allocation with each path of play. An extensive *mechanism* $\Gamma \equiv (N, T, g)$ –or extensive game form– is a game with possibly simultaneous moves.[7]

Let $m \in M$. Let $g(m)$ be the allocation prescribed by the path induced induced by m, and $g_i(m)$ the *ith* component of $g(m)$. Let $g(m, s)$ be the allocation corresponding to m starting at node s. Let $\mathcal{G}$ be the *set of mechanisms.*

For the definitions at hand, suppose that the only characteristics that can vary are the preferences of agents. That is, the set of agents N and the endowment $\bar{\omega}$ are fixed. An economy then reduces to a preference profile $R \in \mathcal{R}^N$.

Let $\Gamma \in \mathcal{G}$. Given $R \in \mathcal{R}^N$, Γ defines a *non-cooperative game* in extensive form (Γ, R). For each $R \in \mathcal{R}^N$, a *game-theoretic solution concept* E (henceforth solution concept) describes a set of predictions on how Γ will be played, as a function of the agents' preferences. It is a mapping $E : \mathcal{G} \times \mathcal{R}^N \twoheadrightarrow 2^M$. For each $R \in \mathcal{R}^N$, let $E(\Gamma, R)$ and $E\mathcal{O}(\Gamma, R)$ be the set of *E-equilibrium* and *E-equilibrium outcomes* of (Γ, R). The definition of solution concepts is indeed very broad and encompasses many concepts in which agents do not behave "strategically". In order to narrow the definition, we introduce the minimal requirement of *responsiveness* of a solution concept.

Responsiveness: E is *responsive* with respect to $\Gamma \in \mathcal{G}$ if for each $\{R, R'\} \subset \mathcal{R}^N$ and each $x \in E\mathcal{O}(\Gamma, R) \setminus E\mathcal{O}(\Gamma, R')$, there exist $i \in N$ and $\{y_i, z_i\} \subset A_i$

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[7] The definition we give encompasses both static and sequential game forms: a static game form can always be represented via an extensive game form with simultaneous moves.

such that,

$$y_i \; R_i \; z_i \text{ and } z_i \; P_i' \; y_i.$$

Observe that each solution concepts commonly used in the implementation literature –such as Nash equilibrium and any of its refinements– is indeed responsive.

A solution f is *implemented* by Γ in E if for each $R \in \mathcal{R}^N$, we have $E\mathcal{O}(\Gamma, R) = f(R)$

Examples of solutions concepts are numerous. To define just a few, a Nash equilibrium of (Γ, R) is a strategy profile $m^* \in M$ such that for each $i \in N$ and each $m_i \in M_i$,

$$g(m^*)_i \; R_i \; g(m_i, m^*_{-i})_i.$$

A subgame perfect equilibrium of (Γ, R) is a strategy profile $m^* \in M$ such that for each $s \in S \setminus Z$, each $i \in N$, and each $m_i \in M_i$,

$$g(m^*, s)_i \; R_i \; g(m_i, m^*_{-i}, s)_i.$$

But solutions concepts are not only limited to Nash equilibrium and its refinements. Solution concepts such as dominant strategies or undominated strategies are also part of our general definition.[8]

Implementability via some game-theoretic solution concept: A solution f satisfies *implementability via some game-theoretic solution concept* if and only if there exist Γ and a solution concept E, responsive with respect to Γ, such that f is implemented by Γ in E.

We close this section and give the definitions of *Maskin monotonicity* (Maskin, 1999), a necessary condition for Nash implementation, and *Unchanged contour independence* (Maniquet, 2003), a sufficient condition for subgame perfect implementation.

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[8] Our definition of *solution concepts* should be seen as encompassing any solution concepts commonly used in the implementation literature since they all share a common feature that will be explained in the next section: in order to break a equilibrium when going from one preference profile to another, a change in preferences is needed.

Maskin monotonicity: For each $\{(R,\bar{\omega}),(R',\bar{\omega})\} \subset E^N$ and each $x \in f(R,\bar{\omega})$,

$$[\text{For each } i \in N,\ LC_i(R_i,x_i) \subseteq LC_i(R'_i,x_i)] \implies [x \in f(R',\bar{\omega})].$$

Unchanged contour independence: For each $\{(R,\bar{\omega}),(R',\bar{\omega})\} \subset E^N$ and each $x \in f(R,\bar{\omega})$,

$$[\text{For each } i \in N,\ I_i(R_i,x_i) \cap A_i(\bar{\omega}) = I_i(R'_i,x_i) \cap A_i(\bar{\omega})] \implies [x \in f(R',\bar{\omega})].$$

3 Robustness of equal-budget choice equivalent rules

Equal-budget choice equivalent allocation rules are indexed by a number α and can be decomposed into three categories that are specific to the region in which α is located[9,10]:

$$\begin{aligned} 1)\ \alpha &\in \left\{0 < \alpha' < \bar{\omega} :\ \alpha' \notin C(A(\bar{\omega})) \text{ and } \alpha' \neq \frac{\bar{\omega}}{n}\right\} \\ 2)\ \alpha &\in C(A(\bar{\omega})), \text{ and} \\ 3)\ \alpha &= \frac{\bar{\omega}}{n}. \end{aligned}$$

When $\alpha = \frac{\bar{\omega}}{n}$, f_α is *equal-budget choice equivalent with respect to* $\mathcal{B}_\alpha$ if and only if it coincides with the *equal-income Walrasian rule.* We show this result in the lemma below.

Lemma 1 (Thomson, 1994) *Let the domain of preferences be* $\mathcal{R}_C^N$. *Consider the family of linear choice sets* $\mathcal{B}_{\frac{\bar{\omega}}{n}}$. *An allocation rule* f_α *is equal-budget choice equivalent if and only if* $f_\alpha = W_E$.

Proof: Let $f_{\frac{\bar{\omega}}{n}}$ be the *equal-budget choice equivalent* allocation rule with respect to $\mathcal{B}_{\frac{\bar{\omega}}{n}}$ and let R, $\bar{\omega}$ and $x \in f_{\frac{\bar{\omega}}{n}}(R,\bar{\omega})$ be given. By definition, there exists $p \in \Delta$ such that for each $i \in N$, $x_i\ I_i\ x_i^*$ for each $x_i^* \in \arg\max R_i|_{B_{\frac{\bar{\omega}}{n}}(p)}$. We claim that for each $i \in N$, $p \cdot x_i = p \cdot \alpha$, i.e. $x_i \in \arg\max R_i|_{B_{\frac{\bar{\omega}}{n}}(p)}$.

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[9] We point out that for any $\alpha \gg \frac{\bar{\omega}}{n}$, the allocation rule f is empty at each preference profile.

[10] We voluntarily exclude the trivial corners $\{0\}$ and $\{\bar{\omega}\}$ as point of division since they both deliver empty solutions.

By definition, for each $i \in N$, we have that $p \cdot x_i \geq p \cdot x_i^*$. Suppose, contrary to our claim, that for some $j \in N$, $p \cdot x_j > p \cdot x_j^*$. Summing across agents, we obtain that $\sum p \cdot x_i > \sum p \cdot x_i^* = p \cdot \frac{\bar{\omega}}{n}$, a contradiction with the feasibility of x as an allocation. Therefore, $f \subset W_E$. The other inclusion is obvious.

Q.E.D.

3.1 Robustness with Respect to Changes in Preferences and Implementation

We start this section by showing a clear connection between *independence to irrelevant preference changes* and the decentralization of a rule. In fact, such a robustness property is the most basic requirement for an allocation rule to be *implementable via some game-theoretic solution concept.* For the remaining of the subsection, we fix the set of agents N and the endowment $\bar{\omega}$. A rule f is then a mapping $f : \mathcal{R}^N \twoheadrightarrow A(\bar{\omega})$. An economy is thus simply a preference profile R and a class of economy is simply a set of preference profiles $\mathcal{R}^N$.

Proposition 1: *A solution f satisfies implementability via some responsive game-theoretic solution concept only if it satisfies independence to irrelevant preference changes.*

Proof: Since f satisfies *implementability via some game-theoretic solution concept*, there exists a mechanism Γ and solution concept E, responsive with respect to Γ, such that f is implemented by Γ in E. Fix a preference profile $R \in \mathcal{R}^N$ and $x \in f(R)$. By *implementability via some game-theoretical solution concept*, there exists $m \in M$ that is an E-equilibrium and such that $g(m) = x \in E\mathcal{O}(\Gamma, R)$.

Consider $R' \in \mathcal{R}^N$, $R' \neq R$, such that for each $i \in N$, $R_i \cap A_i = R'_i \cap A_i$. Since $R' \neq R$, for some $j \in N$, $R_j \neq R'_j$. The region where preferences determined that m was an E-equilibrium, and that $x \in E\mathcal{O}(\Gamma, R)$, still contains the exact same set of preferences as in R. Since E is responsive, we obtain that $x \in E(\Gamma, R')$ and thus $x \in f(R')$.

Q.E.D.

Proposition 1 states an obvious requirement for *implementability via some game theoretic solution concept.* Information on preferences outside of the feasible set cannot be used. The reason is that in order to break an equilibrium when going from one preference profile to another, a change in preferences is needed. For implementation via Nash equilibrium and any of its refinement, specific preference reversals are needed. For instance, implementation via Nash equilibrium requires that if $x \in f(R) \setminus f(R')$, there exists $i \in N$ and $y_i \in A_i$ such that $x_i\ R_i\ y_i$ and $y_i\ P_i'\ x_i$. On the other hand, for implementation via undominated strategies, if $x \in f(R) \setminus f(R')$ there exists $i \in N$ and $y_i \in A_i$ such that either $x_i\ R_i\ y_i$ and $y_i\ P_i'\ x_i$, or $x_i\ P_i\ y_i$ and $x_i\ I_i'\ y_i$.[11] Finally, implementation via subgame perfect equilibrium requires that if $x \in f(R) \setminus f(R')$, there exists $i \in N$, and y_i, $z_i \in A_i$ such that $y_i\ R_i\ z_i$ and $z_i\ P_i\ x_i$.

However preference changes outside of the feasible set cannot be tested since they occur at allocations that are not feasible. An allocation rule that is too sensitive to change in preferences outside of the feasible set may violate implementability.[12]

Next we show that when the domain is $\mathcal{R}_C^N$, the only *equal-budget choice equivalent* rules that are not sensitive to a change in preferences occuring only outside of the feasible set are the rules f_α with $\alpha \in C\left(A\left(\bar{\omega}\right)\right)$. Differentiability plays a crucial rule in the implementation of the *equal-income Walrasian rule*. When indifference curves can have kinks, the *equal-income Walrasian rule* is not implementable via any solution concept. As expected, the problem comes from allocations that are at the boundary of the feasible set but it goes beyond the classical impossibility result on the Nash implementation of this rule.

Proposition 2: *Suppose the class of preferences is $\mathcal{R}_C^N$. An equal-budget choice equivalent allocation rule f_α satisfies implementability if and only if $\alpha \in C(A(\bar{\omega}))$*

Proof: The proof relies on a series of lemmas.

Lemma 1: *Suppose that $\alpha \in \left\{0 < \alpha' < \bar{\omega} :\ \alpha' \notin C(A\left(\bar{\omega}\right)) \text{ and } \alpha' \neq \frac{\bar{\omega}}{n}\right\}$. The equal-budget choice equivalent allocation rule f_α with respect to $\mathcal{B}_\alpha$ does not satisfy implementability.*

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[11] See Jackson (1992) for a definition of implementation in undominated strategies.

[12] Notice also that such allocations rules cannot be virtually implemented either. For a definition of Virtual Nash implementation, see for instance Abreu and Sen (2001) or Bochet and Maniquet (2006).

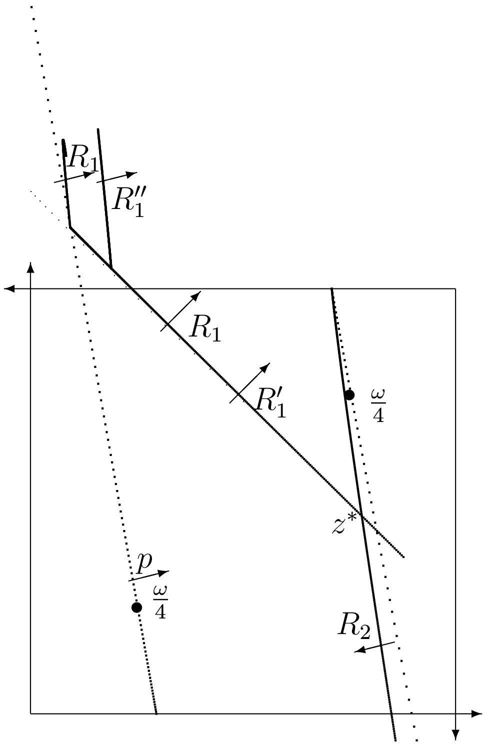

Figure 1

Proof: We use counter-examples with $\alpha = \frac{\bar{\omega}}{K}$ and $K > n$ to prove the claim.[13] The proof generalizes to any $\alpha \in \left\{0 < \alpha' < \bar{\omega} :\ \alpha' \notin C(A(\bar{\omega})) \text{ and } \alpha' \neq \frac{\bar{\omega}}{n}\right\}$.

Let $n = \ell = 2$, $\mathcal{R} \subset \mathcal{R}_C^N$ with $\mathcal{R} = (R, R')$, $R = (R_1, R_2)$, $R' = (R_1', R_2')$ with $R_2 = R_2'$, and $\bar{\omega} = (4, 4)$. The preferences R_1, R_1' and R_2 are represented by the following utility functions as follows.

$u_1(x_1, y_1) = \min\left\{x_1 + y_1;\ x_1 + \frac{1}{K+1}y_1 + \frac{5K}{K+1}\right\}$
$u_1'(x_1, y_1) = x_1 + y_1$
$u_2(x_2, y_2) = (K+1)x_2 + y_2$

For each $K > n$, when the profile is R, there exists an allocation $z^* \in f_{\frac{\bar{\omega}}{K}}(R)$, that is *equal-budget choice equivalent* and for which $(x_1, 5) \in \arg\max R_1|_{B_{\frac{\bar{\omega}}{K}}(p)} \notin A_i$, $x_1 > 0$, for some $i \in N$ and $p = (K, 1)$. The situation is depicted for the case $K = 4$ in Figure 1 above.

There is an *equal-budget choice equivalent* allocation,

$$z^* = \left(\left(\frac{123}{48}, \frac{137}{48}\right), \left(\frac{69}{48}, \frac{55}{48}\right)\right),$$

and $\arg\max R_1|_{B_{\frac{\bar{\omega}}{4}}(p)} = \left(\frac{5}{12}, 5\right)$. However, allocation z^* is no longer *equal-budget choice equivalent* with respect to $B_{\frac{\bar{\omega}}{4}}(p)$ at R': the indifference curve

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[13] The case $K < n$ is trivial. For any $\alpha \gg \frac{\bar{\omega}}{n}$, the equal-budget choice equivalent allocation rule with respect to $\mathcal{B}_\alpha$ is then empty at every preference profiles.

of agent 1 at the point $\left(\frac{5}{12}, 5\right)$ extends below $B_{\frac{\bar{\omega}}{4}}(p)$ and hence $\left(\frac{5}{12}, 5\right) \notin \arg\max R_1'|_{B_{\frac{\bar{\omega}}{4}}(p)} = \left(0, \frac{20}{3}\right)$. Finally, observe that the preferences of agents inside the feasible set have not changed. The correspondence $f_{\frac{\bar{\omega}}{4}}$ is not implementable in **any** game theoretical solution concept. It is clear that the same conclusion applies to any $K > n$.

Q.E.D.

Remark 1: The counter-examples we used were based on an extension of an indifference curve below the set $B_{\frac{\bar{\omega}}{K}}(p)$ for agent 1 when going from R to R'. For each $K > n$, the same conclusion remains true even if there is an expansion of the lower contour set at z^* outside of the feasible set. To see this, suppose that there is also a third preference profile (R_1'', R_2) at which preferences of agent 1 are given by $u_1(\cdot, R_1'') = \min\left\{x_1 + y_1;\ x_1 + \frac{1}{K+1}y_1 + \frac{4.5K}{K+1}\right\}$. As seen in Figure 1, there is an expansion of the lower contour set for agent 1 when going from R to R'' but the expansion occurs outside of the feasible set while the preferences inside the feasible set remain unchanged. This shows that the violation of *implementability* is not restricted to $\mathcal{R}_C^N$ but **also** applies to $\mathcal{R}_D^N$ as well.

Finally, notice that the examples we have used are based on extension of budget sets beyond the upper boundary of the Edgeworth box. For a point $\alpha = (0, k)$ with $k < \bar{\omega}^2$, the extension of the budget sets will occur on the right boundary of the box. The same examples as we have constructed above can be adapted to this case.

Lemma 2: *Suppose the class of preferences is* $\mathcal{R}_C^N$. *The allocation rule* W_E *does not satisfy implementability*

Proof: We adapt the counter example of lemma 1 to the case $K = n = 2$. The aggregate endowment is $\bar{\omega} = (4, 4)$. The preferences of agents are as follows.

$u_1(x_1, y_1) = \min\left\{x_1 + y_1;\ x_1 + \frac{1}{3}y_1 + \frac{8}{3}\right\}$, $u_1'(x_1, y_1) = x_1 + y_1$ and $u_2(x_2, y_2) = 2x_2 + y_2$.

There exists $z^* \in W_E(R)$ with supporitng price p^* that is on the boundary of the feasible set with $z^* = ((1,4); (3,0))$ and $p^* = (\frac{2}{3}, \frac{1}{3})$.

The situation is depicted in Figure 2.

Observe that $z^* \notin W_E(R')$. However, there exists no $i \in N$ and no $\{x_i, y_i\} \subset A_i$ such that

$$x_i\ R_i\ y_i \text{ and } y_i\ P_i'\ x_i.$$

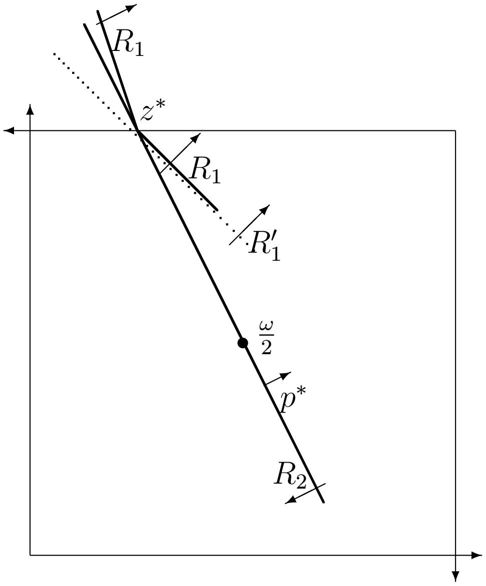

Figure 2

In conclusion, W_E is not implementable in **any** solution concept.[14]
Q.E.D.

Lemma 3: *Suppose that $\alpha \in C(A(\bar{\omega}))$, then f_α satisfies implementability.*

Proof: Clearly, f_α satisfies *independence to irrelevant preference changes.* Observe that for any such α, f_α satisfies *unchanged contour independence*, a sufficient condition for subgame perfect implementation.[15] Given $R \in \mathcal{R}_C^N$,for each $x \in f_\alpha(R)$, there exists $p \in \Delta$ such that $\cap_{i \in N} \{x_i \in B_\alpha(p)\} = \emptyset$, and for each $i \in N$, $x_i \; I_i \; \arg\max R_i|_{B_\alpha(p)} \in A_i$. Consider a profile $R' \in \mathcal{R}_C^N$ such that $I_i(R_i, x_i) \cap A_i = I_i(R'_i, x_i) \cap A_i$ for each $i \in N$. We have that for each $i \in N$, $z_i \in \arg\max R_i|_{B_\alpha(p)} \iff z_i \arg\max R'_i|_{B_\alpha(p)}$. Therefore, $x \in f(R')$.
Q.E.D.

The combination of lemma 1 to 3 completes the proof of Proposition 2.

Proposition 3: *Suppose the class of preferences is $\mathcal{R}_D^N$. An equal-budget choice equivalent allocation rule f_α satisfies implementability if and only if $\alpha \in C(A(\bar{\omega})) \cup \left\{\frac{\bar{\omega}}{n}\right\}$.*

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[14] Obviously, the same is true for any initial distribution of the aggregate endowment different of equal-division: if the domain is $\mathcal{R}^{ND}$, then the *Walrasian correspondence* is not implementable in any solution concept (see Bochet, 2005a).

[15] Notice that if $\alpha \in C(A(\bar{\omega}))$, f_α is not *Maskin monotonic* and hence not Nash implementable.

Proof: We have already shown in lemma 1 that even if preferences are differentiable and $\alpha \in \left\{0 < \alpha' < \bar{\omega} : \alpha' \notin C(A(\bar{\omega})) \text{ and } \alpha' \neq \frac{\bar{\omega}}{n}\right\}$, then f_α cannot satisfy *equal-budget choice equivalence* and *implementability.*

Finally, observe that W_E now satisfies *unchanged contour independence* (see Maniquet (2003) and Bochet (2005a)).[16]

Q.E.D.

Remark 2: Only a few allocation rules satisfy *equal-budget choice equivalence* and *implementability.* By the first welfare theorem, W_E only contains *Pareto efficient* allocations. This is clearly not the case when $\alpha \neq \frac{\bar{\omega}}{n}$. We point out in passing that whenever $\alpha \neq \frac{\bar{\omega}}{n}$, there does not exist a *Pareto efficient* and Nash implementable selection of the *equal-budget choice equivalent* allocation rule f_α.

Moreover, our point of view is that the boundary rules do not offer a sensible approach to fairness. After all, when deciding how to first divide the resources, the preferences of agents are in general unknown to the planner. By discarding some goods, the planner goes against a principle of neutrality with respect to goods. For instance suppose that there are two goods, food and water. The planner should not decide to give water to agents but no food to start with when both items are available. It feels thus natural to impose the familiar *neutrality* requirement defined below.

Neutrality: An allocation rule f_α satisfies *neutrality* if and only if for each permutation $\pi : L \to L$ and each $R \in \mathcal{R}^N$,

$$x \in f_\alpha(R) \implies \pi \circ x \in f_{\pi \circ \alpha}(\pi \circ R).$$

Based on this argument, a strengthening of proposition 3 with neutrality would provide a simple "characterization" of the equal-income Walrasian rule. In addition, it provides a further justification of equal-division of resources as a first step towards fairness.

3.2 Robustness with Respect to Change in the Number of Agents

In this section we go back to the initial setting introduced in section 2, where the number of agents in the economy can vary. We start with the study of

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[16] Obviously, the *equal-income Walrasian allocation rule* is not implementable in dominant strategies, in undominated strategies or in Nash equilibrium.

our second robustness property, *consistency.*[17]

Proposition 4: *Suppose the class of economies is* E_D^N. *An allocation rule* f_α *satisfies consistency if and only if* $\alpha = \frac{\bar{\omega}}{n}$.

Proof:

First part (Thomson, 1988): W_E satisfies consistency

Take an economy $(R, \bar{\omega}) \in E_D^N$ and fix $x \in W_E(R, \bar{\omega})$. By definition, there exists $p \in \Delta$ such that for each $i \in N$, $x_i \in \arg\max R_i|_{B_{\frac{\bar{\omega}}{n}}(p)}$. Given the assumptions on preferences, for each $i \in N$, $p \cdot x_i = p \cdot \frac{\bar{\omega}}{n}$. Consider a subeconomy $M \subset N$. The total resources available in this subeconomy after the departure of agents in $N \backslash M$ with their respective bundle is $\sum_{i \in M} x_i$. Hence, allocation x_M is still feasible. Since $p \cdot x_i = p \cdot \frac{\bar{\omega}}{n}$, we have that $p \cdot \frac{\sum_{j \in M} x_j}{|M|} = p \cdot \frac{\bar{\omega}}{n}$. Therefore, $B_{\frac{\bar{\omega}}{n}}(p) = B_{\frac{\sum_{j \in M} x_j}{|M|}}(p)$ and thus $x_j \in f_{\frac{\sum_{j \in M} x_j}{|M|}}\left(R \backslash R_{N \backslash M},\ \sum_{j \in M} x_j\right)$.

Second part: Let $\alpha \in C(A(\bar{\omega}))$. Then f_α violates consistency

Let us look at the following counter-example.

Let $n = 3$, $\ell = 2$, $R \in \mathcal{R}_D^N$, $\bar{\omega} = (4, 2)$ and $\alpha = (0, \bar{\omega}^y)$. At profile $R \in \mathcal{R}_D^N$, agents's preferences are represented by the following functions: $u_i(x_i, y_i) = \frac{2}{3}x_i + y_i$ for $i = 1, 2$ and $u_3(x_3, y_3) = 3x_3 + y_3$. Consider the *equal-opportunity equivalent* allocation $z = ((1.5, 1); (1.5, 1); (1, 0))$ for $p = (2, 1)$. At these prices, we have $\arg\max R_i|_{B_\alpha(p)} = (0, 2)$ for $i = 1, 2$, $\arg\max R_3|_{B_\alpha(p)} = (1, 0)$ and $\arg\max R_j|_{B_\alpha(p)}\ I_i\ z_j$ for each $j = 1, 2, 3$. Suppose that agent 1 leaves the economy with z_1. The reduced economy has $\bar{\omega} = (2.5, 1)$ and $\alpha = (0, 1)$. By *consistency*, $((1.5, 1); (1, 0)) \in f_{(0,1)}(R \backslash R_1,\ (2.5, 1))$. The only price vector such that $\arg\max R_3|_{B_{(0,1)}(p)} = (1, 0)$ is $p = (1, 1)$. At this prices, $\arg\max R_2|_{B_{(0,1)}(p)} = (0, 1)$ but $z_2\ P_i\ \arg\max R_2|_{B_{(0,1)}(p)}$, a contradiction with the fact that f_α satisfies *equal-budget choice equivalence.*

Third part: Let $\alpha \notin C(A(\bar{\omega})) \cup \left\{\frac{\bar{\omega}}{n}\right\}$. Then f_α violates *consistency.*

We use one counter-example but it should be clear that the same can be shown for any other $\alpha \notin C(A(\bar{\omega})) \cup \left\{\frac{\bar{\omega}}{n}\right\}$.

Let $n = 3$, $\ell = 2$, $R \in \mathcal{R}_D^N$, $\bar{\omega} = (5, 2)$ and $\alpha = (0.5, 2)$. At profile $R \in \mathcal{R}_D^N$, agents's preferences are represented by the following functions: $u_i = x_i + y_i$

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[17] Thomson (1988) showed that if indifference curves have kinks, then W_E does not satisfy consistency. For the remainder of the paper, the class of economies we consider is E_D^N.

for $i = 1, 2$ and $u_3 = (x_3)^2 y_3$. Allocation $z = ((2, 0.5); (2, 0.5); (1, 1))$ is *equal-opportunity equivalent* with respect to $\mathcal{B}_\alpha$ for $p = (2, 1)$. Suppose now that agent 1 leaves with z_1. The reduced economy has $\bar{\omega} = (3, 1.5)$ and $\alpha = (0.3, 1.5)$. By consistency, $((2, 0.5); (1, 1)) \in f_{(0.3,1.5)} (R \backslash R_1,\ (3, 1.5))$. The only price vector such that $\arg\max R_2|_{B_{(0.3,1.5)}(p)}\ I_2\ (2, 0.5)$ is $p = \left(\frac{15}{22}, 1\right)$. However, $\arg\max R_3|_{B_{(0.3,1.5)}\left(\frac{15}{2},1\right)}\ P_3\ z_3$, a contradiction with *equal-budget choice equivalence.*

We conclude that the proposition holds if and only if $\alpha = \frac{\bar{\omega}}{n}$.

Q.E.D.

We conclude our results by the study of *replication invariance.* We provide a characterization that parallels a result of Thomson (1988). He shows that a selection from the *Pareto efficient* and *equal-split lower bound* correspondence satisfies *consistency* and *replication invariance* if and only if it is a subrule of the *equal-income Walrasian rule.*

Proposition 5: *Suppose the class of preferences is $\mathcal{R}_D^N$ and fix $\alpha \in \mathbb{R}_+^\ell$. A Pareto efficient selection from f_α satisfies consistency and replication invariance if and only if $\alpha = \frac{\bar{\omega}}{n}$.*

Proof: Let $\alpha = \alpha_{\bar{\omega}} \cdot \bar{\omega} \in \mathbb{R}_+^\ell$, $(R, \bar{\omega}) \in E_D^N$ and $x \in f_{\alpha_{\bar{\omega}} \cdot \bar{\omega}} (R, \bar{\omega})$ be given. By *Pareto efficiency*, there exists $p \in \Delta$ that supports x. Suppose that $x \notin W_E (R, \bar{\omega})$. Then there exists $i \in N$ with $p \cdot x_i > p \cdot \frac{\bar{\omega}}{n}$ and $j \in N$ with $p \cdot x_j < p \cdot \frac{\bar{\omega}}{n}$. By convexity and monotonicity of preferences, there exists λ_i,$\lambda_j \in \mathbb{N}$ and y such that $y\ P_j\ x_j$, and

$$y = \frac{\lambda_i x_i + \lambda_j x_j}{\lambda_i + \lambda_j}.$$

Define $\lambda \equiv \max \{\lambda_i, \lambda_j\}$. By *replication invariance*, $\lambda x \in f_{\alpha_{\bar{\omega}} \cdot \lambda\bar{\omega}} (\lambda R, \lambda\bar{\omega})$. Consider a subeconomy of λR composed of λ_i agents of type i and λ_j agents of type j. By *consistency*, we obtain that $(\lambda x_i, \lambda x_j) \in f_{\alpha_{\bar{\omega}}(\lambda_i+\lambda_j)\lambda y} (\lambda_i R_i, \lambda_j R_j, \lambda (\lambda_i + \lambda_j) y)$. By *equal-opportunity equivalence*, there exists $p \in \Delta$ such that $\lambda x_k\ I_k\ \arg\max R_k|_{B_{\lambda y}(p)}$ $R_k\ \alpha_{\bar{\omega}}(\lambda_i + \lambda_j)\lambda y$ for $k = i, j$. This implies that $\lambda x_j\ R_j\ \alpha_{\bar{\omega}}(\lambda_i + \lambda_j)\lambda y$, and $\alpha_{\bar{\omega}} (\lambda_i + \lambda_j) \lambda x_j\ P_j\ \alpha_{\bar{\omega}}(\lambda_i + \lambda_j)\lambda y$, a contradiction with the initial assumption that $y\ P_j\ x_j$.

Q.E.D.

4 Conclusion

We conclude the paper with two remarks.

Remark 3: As seen in lemma 2, for some preference domains, the *equal-income Walrasian correspondence* is not implementable via any game-theoretical solution concept. On the other hand, a correspondence called the *constrained equal-income Walrasian correspondence*, CW_E, is Nash implementable. Given the deficits of the allocation rules we have studied, one can alternatively look at a constrained version of *equal-budget choice equivalence.*

Given $\alpha \in \left\{0 < \alpha' < \bar{\omega} : \ \alpha' \notin C(A(\bar{\omega})) \text{ and } \alpha' \neq \frac{\bar{\omega}}{n}\right\}$ and $p \in \Delta$, a constrained linear choice set is $\bar{B}_\alpha(p) \equiv \{x \in A_i(\bar{\omega}) : \ p \cdot x \leq p \cdot \alpha\}$. Based on this definition, we can reformulate our axiom in term of *constrained equal-opportunity equivalence* and *constrained equal-budget choice equivalence.*

Constrained equal-opportunity equivalence: An allocation $x \in A(\bar{\omega})$ is *constrained equal-opportunity equivalent relative to the family $\mathcal{B}_\alpha$ for $R \in \mathcal{R}^N$* if there exists $\bar{B}_\alpha(p) \in \mathcal{B}_\alpha$ such that for each $i \in N$, $x_i \ I_i \ x_i^*$ for some $x_i^* \in \arg\max R_i|_{\bar{B}_\alpha(p)}$.

The definition of *equal-budget choice equivalence* allocation rule $\bar{f}_\alpha$ then just follows. When $\alpha \in \left\{0 < \alpha' < \bar{\omega} : \ \alpha' \notin C(A(\bar{\omega})) \text{ and } \alpha' \neq \frac{\bar{\omega}}{n}\right\}$, every $\bar{f}_\alpha$ is subgame perfect implementable. However, notice that the only Nash implementable allocation rule is the *constrained equal-income Walrasian rule* CW_E.[18] Therefore, an allocation rule $\bar{f}_\alpha$ satisfies *constrained equal-budget choice equivalence* and *Maskin monotonicity* if and only if $\bar{f}_\alpha = CW_E$.[19] Observe that $WE_E \subseteq CW_E$. In addition, if preferences are strongly monotonic, then for any $(R, \bar{\omega}) \in E^N$ and any $x \in CW_E(R, \bar{\omega})$, allocation x is *Pareto efficient.* An important observation is that if $\alpha \neq \frac{\bar{\omega}}{n}$, then $\bar{f}_\alpha$ is neither a subcorrespondence nor a supercorrespondence of its unconstrained counterpart f_α.

Remark 4: An interesting extension of the work presented here would be to consider fair solutions suggested in the literature and to construct the family of choice sets generating them as *equal-opportunity equivalent* solutions.

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[18] Thomson (1999) in fact shows that CW_{EQ} is the minimal Maskin monotonic extension of W_{EQ}.

[19] Observe that a selection of the Constrained equal-income Walrasian rule is not necessarily Maskin monotonic.

If they satisfy *implementation via some game theoretic solution concept*, then the components of the family of choice sets (e.g. bundles, allocations etc...) can be used as part of the strategy sets of agents. For instance, suppose that a given solution is Nash implementable using a family of choice sets $\mathcal{B}$. Then, we conjecture that the announcements of preference profiles in canonical mechanisms such as Abreu and Sen (1990) for subgame perfect implementation; Maskin (1999) or Bochet (2005b) for Nash implementation; can be replaced by announcements of choice sets. This work would suggest a close connection between *equal-opportunity equivalence* and the possibility to perform strategy space reductions from infinite to finite spaces in canonical mechanisms used to implement an allocation rule in some responsive solution concept.

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