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**Economics of Uncertainty and Information**
**Homework 2**

Note: There will be a 3rd homework posted on my webpage, most probably on the last day of class. The topic will be adverse selection.

**Exercise 1**: Consider the following exchange economy under incomplete information.

The set of agents is $N = \{1, 2, 3, 4\}$. There is only one good. Each agent $i$ is initially endowed with $\omega_i = 1$ unit of the good, and this irrespective of the type profile. The aggregate endowment is $\bar{\omega}$.

We have that $T_i = \{t_i, t'_i, t''_i\}$ $i = 1, 2$ and $T_j = \{t_j, t'_j\}$ $j = 3, 4$. The set of possible state of the world is $T$. However, the set of states occurring with positive probability is $T^\star \subset T$, where

$$T^\star = \{t, t', t''\}$$

and,

$$\begin{aligned} t &= (t_1, t_2, t_3, t_4) \\ t' &= (t'_1, t'_2, t'_3, t'_4) \\ t'' &= (t''_1, t''_2, t'_3, t'_4) \end{aligned}$$

For agents 3 and 4, the conditional probabilities are as follows,

$$q_j(s \left| t'_j \right.) = \frac{1}{2} \text{ for } s = t', t'' \text{ and } j = 3, 4.$$

An allocation $x$ is a redistribution of the aggregate endowment, a state contingent bundle $x_i$ for each agent $i \in N$, where $x_i = (x_i(t), x_i(t'), x_i(t''))$.

The set of (feasible) allocations is state independent and is denoted $A$. Formally,

$$A = \{x \in \mathbb{R}^4 : \sum_i x_i \leq \bar{\omega}\}$$

Agents 1 and 2 have a utility function that is state independent and given by

$$u_i(x_i(s), s) = x_i(s) \text{ for each } s \in T^\star, \text{ for } i = 1, 2.$$

Agent 3 has the following state dependent utility function,

$$\begin{aligned} u_3(x_3(t), t) &= x_3(t) \\ u_3(x_3(t'), t') &= \sqrt{x_3(t')} \\ u_3(x_3(t''), t'') &= \sqrt{4x_3(t'')} \end{aligned}$$

On the other hand, agent 4 differs from agent 3 as follows,

$$\begin{aligned} u_4(x_4(t), t) &= x_4(t) \\ u_4(x_4(t'), t') &= \sqrt{4x_4(t')} \\ u_4(x_4(t''), t'') &= \sqrt{x_4(t'')} \end{aligned}$$

An allocation $x$ is ex-post individually rational if for each $i \in N$

$$u_i(x_i(s), s) \geq u_i(\omega_i, s) \ \forall s \in T^\star.$$

An allocation $x$ is interim individually rational if for each agent $i \in N$,

$$\sum_{s \in T^\star} q_i(s\,|t_i)u_i(x_i(s), s) \geq \sum_{s \in T^\star} q_i(s\,|t_i)u_i(\omega_i, s) \ \forall t_i \in T_i$$

1) What are the conditional probabilities over states for agent 1 and 2? Explain.

2) Show that any allocation in this economy is incentive compatible. Explain.

3) Is the no-trade allocation ex-post efficient and ex-post individually rational?

4) Can you suggest an allocation that interim dominates the no-trade allocation and that is interim individually rational? Is this allocation ex-post individually rational? Comment.

5) Can you say something about the set of interim efficient and individually rational allocations?

6) Suppose we eliminate agent 1 from this problem (i.e. $N = \{2, 3, 4\}$). Is it still true that every allocation is incentive compatible?

**Exercise 2**: Consider an exchange economy with $N = \{1, ..., n\}$ agents, $\ell$ goods indexed $y_1, ..., y_l$ and a set of type profiles $T$ where each $t \in T$ occurs with positive probability. Prior probabilities over type profiles are common to all agents and are given by $q(t)$ for each $t \in T$. An allocation is $x = (x(t), x(t'), ...) = (x(t))_{t \in T}$. For each state $t \in T$, a bundle for agent $i \in N$ is $x_i(t) = (y_{i1}(t), ..., y_{i\ell}(t))$. Suppose that each agent $i \in N$ has a state-dependent utility function that is strictly increasing in every good. That is, for each $i \in N$,

$$\frac{\partial u_i(x_i(s), s)}{y_k} > 0 \text{ for each } s \in T \text{ and each } k = 1, ..., \ell.$$

Furthermore, assume that individual endowments are state independent, that is, for each $i \in N$,

$$\omega_i(s) = \omega_i \ \forall s \in T.$$

1) Show that in this situation (unlike the example we saw in class), there does exist an allocation that is ex-post efficient and incentive compatible, i.e. that

$$\Delta_p \cap \Delta^\star \neq \emptyset$$

2) Construct an example where this conclusion remain true if utility functions are **weakly** increasing in every good? To answer this part, you can use an edgeworth box with 2 agents-2 goods.

**Exercise 3**: Consider the following exchange economy under incomplete information. There are two agents 1 and 2, and two goods $y$ and $z$. An allocation is $x$. The aggregate endowment is $\bar{\omega} = (2, 2)$. Agent 1 is of two possible types $t_1$ and $t_1'$, while agent 2 has only one type $t_2$.

$$T = T^\star = \{t = (t_1, t_2), t' = (t_1', t_2)\}$$

Agent 2's utility function is,

$$u_2(x_2(s), s) = y_2 \text{ for all } s \in T$$

For agent 1,

$$\begin{aligned} u_1(x_1(t), t) &= z_1, \\ u_1(x_1(t'), t') &= y_1 z_1 \end{aligned}$$

1) Characterize the set of ex-post efficient allocations. Are all ex-post efficient allocations incentive compatible?

2) Consider the allocation $x = (x(t), x(t'))$, where

$$\begin{aligned} x(t) &= ((y_1(t), z_1(t)), (y_2(t), z_2(t))) = ((2, 2), (0, 0)) \\ x(t') &= ((y_1(t'), z_1(t')), y_2(t'), z_2(t')) = ((0, 2), (2, 0)) \end{aligned}$$

Is this allocation incentive compatible?