Comment on:

Evaluating causal relations in neural systems: Granger causality, directed transfer function and statistical assessment of significance

by Kaminski et al.

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The directed transfer function (DTF) introduced by Kamiński and Blinowska (1991) is a well-known frequency-domain based measure for the interrelationships in multivariate time series. In the paper by Kamiński et al. (2001), the authors claim a relationship between the DTF and the concept of Granger causality. Here, Granger causality from one channel X_i to another channel X_j is defined in terms of a bivariate VAR model

$$X_i(t) = \sum_{u=1}^{p} A_{ii}(u) X_i(t-u) + \sum_{u=1}^{p} A_{ij}(u) X_j(t-u) + e_i(t)$$
$$X_j(t) = \sum_{u=1}^{p} A_{ji}(u) X_i(t-u) + \sum_{u=1}^{p} A_{jj}(u) X_j(t-u) + e_j(t),$$

and X_i is said to Granger cause X_j if $A_{ji}(u)$ is nonzero for some $u = 1, \ldots, p$. We note that this bivariate notion of Granger causality has been widely used (e.g., Florens and Mouchart 1985, Goebel et al. 2003, Hesse et al. 2003), but for multivariate systems a more general notion of Granger causality in terms of multivariate VAR models exists (e.g., Sims 1980, Hsiao 1982, Toda and Philipps 1993, Hayo 1999, Eichler 2007, 2005), which is more in line with the original definition by Granger (1969, 1980, 1988).

For the proof of a relation between bivariate Granger causality and DTF, the authors derive the bivariate autoregressive representation of two components of a multivariate VAR(p) process (cf eqs (12) to (14)). We note that the autoregressive representation of a weakly stationary process is defined in terms of linear projections, which implies that the error process $e(t) = (e_i(t), e_j(t))'$ is white noise, that is, the errors at different time points are uncorrelated. In the frequency domain, this implies that the spectral matrix of the error process is constant and equal to $\mathbf{\Sigma}/2\pi$, where $\mathbf{\Sigma} = \mathrm{var}(e(t))$.

In the paper, the authors derive the bivariate autoregressive representation (setting $i = 1$ and $j = 2$) expressed in the frequency domain

$$\left[\mathbf{A}_{11}(\lambda) - \mathbf{A}_{12}(\lambda)\mathbf{A}_{22}(\lambda)^{-1}\mathbf{A}_{21}(\lambda)\right] \begin{pmatrix} X_1(\lambda) \\ X_2(\lambda) \end{pmatrix} = \begin{pmatrix} E_1'(\lambda) \\ E_2'(\lambda) \end{pmatrix} \tag{1}$$

(cf eqn (14)) with error process

$$\begin{pmatrix} E_1'(\lambda) \\ E_2'(\lambda) \end{pmatrix} = \begin{pmatrix} E_1(\lambda) \\ E_2(\lambda) \end{pmatrix} - \mathbf{A}_{12}(\lambda)\mathbf{A}_{22}(\lambda)^{-1} \begin{pmatrix} E_3(\lambda) \\ \vdots \\ E_p(\lambda) \end{pmatrix}.$$

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The spectral matrix $f_{e'_1 e'_2}(\lambda)$ of the error $e'(t)$ process is given by

$$2\pi\,\mathbf{f}_{e'_1 e'_2}(\lambda) = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix} - \mathbf{A}_{12}(\lambda)\mathbf{A}_{22}(\lambda)^{-1}\begin{pmatrix}\Sigma_{31} & \Sigma_{32}\\ \vdots & \vdots\\ \Sigma_{p1} & \Sigma_{p2}\end{pmatrix}$$
$$- \begin{pmatrix}\Sigma_{13} & \dots & \Sigma_{1p}\\ \Sigma_{23} & \dots & \Sigma_{2p}\end{pmatrix}(\mathbf{A}_{22}(\lambda)')^{-1}\mathbf{A}_{12}(\lambda)'$$
$$+ \mathbf{A}_{12}(\lambda)\mathbf{A}_{22}(\lambda)^{-1}\begin{pmatrix}\Sigma_{33} & \dots & \Sigma_{3p}\\ \vdots & \ddots & \vdots\\ \Sigma_{p3} & \dots & \Sigma_{pp}\end{pmatrix}(\mathbf{A}_{22}(\lambda)')^{-1}\mathbf{A}_{12}(\lambda)'$$

Due to the frequency dependency of $\mathbf{A}_{11}(\lambda)$, $\mathbf{A}_{12}(\lambda)$, and $\mathbf{A}_{22}(\lambda)$, this expression in general will not be constant over frequency and, thus, cannot be the spectral matrix of a white noise process. Consequently, the process $e'(t) = (e_1(t), e_2(t))'$ defined by $E'(\lambda) = (E'_1(\lambda), E'_2(\lambda))$ in general is not a white noise process and (1) is not the desired bivariate autoregressive representation.

That $e'(t) = (e_1(t), e_2(t))'$ indeed is not generally a white noise process can be shown by a simple example. Consider a simple trivariate VAR(1) model

$$\begin{aligned}X_1(t) &= \alpha\, X_3(t-2) + \varepsilon_1(t),\\ X_2(t) &= \beta\, X_3(t-1) + \varepsilon_2(t),\\ X_3(t) &= \varepsilon_3(t),\end{aligned}$$

where $\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t))$ is a white noise process with mean zero and variance equal to the identity matrix. On the one hand, we have

$$\mathbf{A}(\lambda) = \begin{pmatrix}1 & 0 & -\alpha\\ 0 & 1 & -\beta\\ 0 & 0 & 1\end{pmatrix},$$

and simple manipulations show that

$$\mathbf{H}(\lambda) = \mathbf{A}(\lambda)^{-1} = \begin{pmatrix}1 & 0 & \alpha\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix},$$

which implies that the DTF from channel 2 to channel 1 is zero.

On the other hand, the bivariate autoregressive representation is given by the best predictor of $\tilde{X}(t) = (X_1(t), X_2(t))$ based on $\tilde{X}(t-1), \tilde{X}(t-2), \ldots$. It can be shown that it is given by

$$\begin{aligned}X_1(t) &= \frac{\alpha\,\beta}{1+\beta^2}\, X_2(t-1) + \tilde{\varepsilon}_1(t),\\ X_2(t) &= \tilde{\varepsilon}_2(t),\end{aligned}$$

where $\tilde{\varepsilon}_2(t) = \varepsilon_2(t) + \beta\,\varepsilon_3(t-1)$ and

$$\tilde{\varepsilon}_1(t) = \varepsilon_1(t) - \frac{\alpha\beta}{1+\beta^2}\,\varepsilon_2(t-1) + \frac{\alpha}{1+\beta^2}\,\varepsilon_3(t-2).$$

Note that $\tilde{\varepsilon}(t) = (\tilde{\varepsilon}_1(t), \tilde{\varepsilon}_2(t))$ is indeed a white noise process satisfying

$$\mathbb{E}\big(\tilde{\varepsilon}(t)\tilde{\varepsilon}(s)'\big) = 0$$

for all $t \neq s$. In particular, we have

$$\operatorname{cov}(\tilde{\varepsilon}_1(t-1), \tilde{\varepsilon}_2(t)) = -\frac{\alpha\beta}{1+\beta^2} + \frac{\alpha\beta}{1+\beta^2} = 0.$$

It follows that X_2 bivariately Granger causes X_1 despite the fact that the DTF is zero. Thus the example contradicts the result by Kamiński et al..

We note that the error process $\tilde{\varepsilon}$ in the above bivariate representation differs from the error process ε' proposed by Kamiński et al., which is of the form (written in the time domain)

$$\begin{aligned} \varepsilon_1'(t) &= \varepsilon_1(t) + \alpha\varepsilon_3(t-1) \\ \varepsilon_2'(t) &= \varepsilon_2(t) + \beta\varepsilon_3(t-2) \end{aligned}.$$

Obviously we have

$$\mathbb{E}(\varepsilon_1'(t-1)\varepsilon_2'(t)) = \alpha\beta \neq 0,$$

that is, the process $\varepsilon'(t) = (\varepsilon_1'(t), \varepsilon_2'(t))$ is not a white noise process as required by the autoregressive representation used in the definition of Granger-causality. As a consequence, the temporal dependence structure that is still hidden in the dependencies of ε' is neglected when computing Granger-causality based on the bivariate representation (1).

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