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Comparing degrees of inequality aversion

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Abstract We propose a straightforward dominance procedure for comparing social welfare orderings (SWOs) with respect to the degree of inequality aversion they express. Three versions of the procedure are considered, each of which uses a different underlying criterion of inequality comparisons: (i) a concept based on the Lorenz quasi-ordering, which we argue to be the ideal version, (ii) a concept based on a minimalist criterion of inequality, and (iii) a concept based on the relative differentials quasi-ordering. It turns out that the traditional Arrow–Pratt approach is equivalent to the latter two concepts for important classes of SWOs, but that it is profoundly inconsistent with the Lorenz-based concept. With respect to the problem of combining extreme inequality aversion and monotonicity, concepts (ii) and (iii) identify as extremely inequality averse a class of SWOs that includes leximin as a special case, whereas the Lorenz-based concept (i) concludes that extreme inequality aversion and monotonicity are incompatible.

1 Introduction

How should we compare different social preference relations over income distributions with respect to the degree of inequality aversion, i.e., the degree of dislike towards inequality, they express? We propose a procedure for comparing degrees of inequality aversion that can be loosely formulated as follows:

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Procedure (⋆) A social welfare ordering (SWO) R is said to be *at least as inequality averse as* an SWO R' if, for all income distributions x and y such that x is *less unequal than* y according to a pre-specified inequality quasi-ordering, (i) R strictly prefers x to y ($x\,P\,y$) whenever R' strictly prefers x to y ($x\,P'\,y$), and (ii) R weakly prefers x to y ($x\,I\,y$ or $x\,P\,y$) whenever R' is indifferent between x and y ($x\,I'\,y$).

In order to make this procedure operational, an inequality quasi-ordering must first be chosen. This feature of Procedure (⋆) makes explicit the fact that, underlying any concept for comparing degrees of inequality aversion, there necessarily has to be a criterion for making comparisons according to inequality—obviously, to be able to check whether an SWO expresses more or less *dislike towards inequality* than another SWO, it must be clear what is meant by inequality in the first place. Once an inequality quasi-ordering is chosen, Procedure (⋆) turns into a fully operational concept of inequality aversion which entails a conceptually straightforward check for dominance: an SWO is referred to as at least as inequality averse as another if it implies, in all relevant choice situations (i.e., those pairs of income distributions that are strictly ranked using the chosen inequality quasi-ordering), an at least as inequality averse choice as the other [as defined in (i) and (ii) of Procedure (⋆)]. Procedure (⋆) can furthermore be shown to be consistent with the common approach of measuring the degree of inequality aversion by the amount of mean income an SWO is prepared to forego in exchange for a given decrease in inequality (see Sect. 3).

Interestingly, the traditional Arrow–Pratt concept for comparing degrees of inequality aversion[1] is a special case of Procedure (⋆). Roughly speaking, the Arrow–Pratt concept of inequality aversion is obtained in the case where the chosen inequality quasi-ordering is the extremely simplistic one which allows only (strict) inequality comparisons between, on the one hand, unequal income distributions and, on the other hand, perfectly equal ones (see Sect. 4). In this article, we take the point of view that while Procedure (⋆) is the appropriate way to approach the problem of comparing degrees of inequality aversion, the Arrow–Pratt version of the procedure is unattractive because it is based on an unduly restrictive inequality quasi-ordering. Taking into consideration its central place in the literature on inequality measurement, the Lorenz inequality quasi-ordering seems a much more suitable candidate for this role. This critique of the Arrow–Pratt concept echoes that of Ross (1981) in the context of decision under risk. Ross argues that for a comparison of risk aversion between two expected utility maximizers, it is not sufficient to compare the premia they are maximally prepared to pay for an insurance against all risks, as the Arrow–Pratt concept prescribes, but it is also necessary to consider premia for insurances that decrease risk to a lower, but still risky, level. Our proposal to consider the concept of inequality aversion based on Procedure (⋆) using the Lorenz

[1] The Arrow–Pratt approach is discussed thoroughly in Lambert (2001).

inequality quasi-ordering is similar to that proposed by Ross since his criterion of decreasing risk is close to the Lorenz criterion.

Throughout the article, we will often be concerned with comparing results yielded by, on the one hand, the version of Procedure ($\star$) that is equivalent to the Arrow–Pratt concept and, on the other hand, the favoured version of Procedure ($\star$) that uses the Lorenz inequality quasi-ordering. It is interesting, however, to consider also a third concept that is intermediate between the Arrow–Pratt concept and the Lorenz-based concept. This third concept is based on the relative differentials quasi-ordering, an inequality criterion that is stronger than the minimalist inequality criterion underlying the Arrow–Pratt concept and weaker than the Lorenz quasi-ordering (see Moyes 1994). Henceforth, we refer to the inequality aversion concept obtained from Procedure ($\star$) using the Lorenz quasi-ordering as the "L-concept," and to that obtained from the procedure using the relative differentials quasi-ordering as the "RD-concept."

We first compare the three concepts of inequality aversion for the class of continuous and monotonic SWOs, the broadest class of SWOs to which the conventional Arrow–Pratt concept is commonly applied. We show that the RD-concept yields the same results as the Arrow–Pratt concept if SWOs are in addition separable, but not necessarily otherwise. Unfortunately, such consistency turns out not to hold between the L-concept and the Arrow–Pratt concept, not even with respect to the important class of constant elasticity of substitution (CES) SWOs (a subclass of the class of continuous, monotonic, and separable SWOs). Usually, a CES SWO with a higher value of the single parameter, ε, is considered more inequality averse than one with a lower value of ε. This role of ε as a measure for the degree of inequality aversion is justified in the framework of the Arrow–Pratt concept of inequality aversion. However, as straightforward examples show, this role of ε is *not* justified if the L-concept is adopted: given two income distributions such that one is less unequal than the other according to the Lorenz inequality quasi-ordering, it is quite possible that a CES SWO with ε strictly prefers the less unequal income distribution, while a CES SWO with $\varepsilon' > \varepsilon$ strictly prefers the more unequal one. Moreover, using a result by Ross (1981) we show that such examples can be found for any two CES SWOs. In other words, if the L-concept is adopted, then no two CES SWOs can be compared with respect to degree of inequality aversion.

Second, we examine the problem of reconciling the ideals of "extreme inequality aversion" and monotonicity, i.e., the question of how to implement the egalitarian ideal of always choosing for less inequality, except in cases where no individual would gain by doing so. We call an SWO extremely inequality averse in a class of SWOs $\mathscr{S}$ if it is at least as inequality averse as all SWOs in $\mathscr{S}$ (and, moreover, is itself a member of $\mathscr{S}$).[2] In the literature, leximin is often seen as a typical example of an SWO that combines extreme inequality aversion with monotonicity. We show that, in the class of monotonic SWOs,

[2] The qualification "extreme" should not be interpreted as a judgment on the ideal of egalitarianism: the term simply stresses the technical point that we are dealing with the most inequality averse position one can take.

both the Arrow–Pratt concept and the *RD*-concept identify the entire class of weakly maximin SWOs as extremely inequality averse—an SWO is said to be *weakly maximin* if it implies a strict preference for a given income distribution over another whenever the worst off is strictly better off in the given income distribution. The class includes leximin and, by consequence, the Arrow–Pratt concept and the *RD*-concept can be said to support the conventional view (see also Tungodden and Vallentyne 2005). However, if the *L*-concept is adopted, this view has to be abandoned: we show that in this case the set of extremely inequality averse monotonic SWOs is empty. Finally, we demonstrate that the incompatibility between extreme inequality aversion and monotonicity is robust with respect to certain reasonable changes in the definition of the idea of extreme inequality aversion.

The article is structured as follows. Section 2 deals with preliminaries. In Sect. 3 we formally introduce and discuss the three concepts for comparing degrees of inequality aversion that constitute the topic of the article. The questions of how the three concepts compare with respect to the class of continuous and monotonic SWOs, and with respect to the idea of extreme inequality aversion, are dealt with in Sects. 4 and 5, respectively. Some concluding remarks are given in Sect. 6.

2 Preliminaries

An *income distribution* is a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{++}$ where $n \geq 3$ is the (fixed) number of individuals in society and x_i is the income of individual i. The set of individuals is N and the set of income distributions is X. We assume that, for all income distributions $x \in X$, individuals are indexed such that $x_1 \leq x_2 \leq \cdots \leq x_n$. In accordance with this assumption, we assume that all considered concepts for welfare and inequality comparisons satisfy *anonymity*—that is, each income distribution is treated equivalently as all of its rearrangements. The arithmetic mean of an income distribution $x \in X$ is written as $\mu(x)$. We use the symbol 1_n to denote an n-dimensional vector of which all components are equal to 1. For a pair of income distributions, $x, y \in X$, we write $x > y$ if $x_i \geq y_i$ for all $i \in N$ with at least one strict inequality, we write $x \not> y$ if $x > y$ does not hold, and we write $x \gg y$ if $x_i > y_i$ for all $i \in N$.

Social preferences are represented by a *social welfare ordering* (SWO) R ("is at least as good as") on X.[3] The asymmetric and symmetric parts of R are denoted by P ("is better than") and I ("is equally good as"), respectively. A *social welfare function* is a function $W : X \to \mathbb{R}$ that represents an SWO.

We now consider some of the axioms used in our analysis. Roughly speaking, continuity ensures that small changes in an income distribution cause only small changes in its social welfare ranking against other income distributions.

Continuity For all $x \in X$, $\{y \,|\, y \in X, y\,R\,x\}$ and $\{y \,|\, y \in X, x\,R\,y\}$ are closed in X.

[3] An ordering is a reflexive, transitive, and complete binary relation.

Monotonicity says that it is an improvement if some individuals get better off without any individuals getting worse off.

Monotonicity For all $x, y \in X$, if $x > y$, then $x \, P \, y$.

Separability requires that the social welfare ranking of a pair of income distributions is not influenced by the incomes that are the same in both income distributions.

Separability For all $\hat{N} \subset N$ and for all $x, y, x', y' \in X$, if $x_i = y_i$ and $x'_i = y'_i$ for all $i \in \hat{N}$, and $x_i = x'_i$ and $y_i = y'_i$ for all $i \in N \setminus \hat{N}$, then $x \, R \, y \Leftrightarrow x' \, R \, y'$.

An SWO that satisfies continuity, monotonicity, and separability can be represented by a social welfare function of the form

$$W : X \longrightarrow \mathbb{R} : x \longmapsto u(x_1) + u(x_2) + \cdots + u(x_n), \tag{1}$$

where $u : \mathbb{R}_{++} \to \mathbb{R}$ is a continuous and strictly increasing function, referred to as a *utility function* (see Bossert and Weymark 2004, Theorem 13.5). We shall pay special attention in our analysis to the constant elasticity of substitution (CES) class of SWOs, an important subclass of the class of continuous, monotonic, and separable SWOs. An SWO R_ε is said to be a member of the CES class if there exists a nonnegative scalar ε such that R_ε can be represented by (1) with utility function $u : t \mapsto (t^{1-\varepsilon})/(1-\varepsilon)$.

Since comparisons of income distributions *with respect to inequality* are conceptually prior to comparisons of SWOs *with respect to degree of inequality aversion*, we require the concept of an *inequality quasi-ordering* (IQO) $\preceq$ ("is at most as unequal as") on X.[4] The asymmetric and symmetric parts of $\preceq$ are denoted by $\prec$ ("is less unequal than") and $\sim$ ("is equally unequal as"), respectively. An *inequality measure* is a function $J : X \to \mathbb{R}$ that represents a complete IQO. The strongest IQO to receive broad acceptance among economists is the Lorenz IQO. The *Lorenz IQO*, written as $\preceq_L$, is defined as follows: for all $x, y \in X$,

$$x \preceq_L y \Leftrightarrow \frac{x_1 + x_2 + \cdots + x_k}{\mu(x)} \geq \frac{y_1 + y_2 + \cdots + y_k}{\mu(y)} \quad \text{for all } k = 1, 2, \ldots, n-1.$$ [5]

An IQO $\preceq$ will be referred to as *Lorenz consistent* if it agrees with all comparisons made by the Lorenz IQO, i.e., if $\prec_L \subset \prec$ and $\sim_L \subset \sim$. We refer to an SWO

[4] A quasi-ordering is a reflexive and transitive binary relation.

[5] For an axiomatic underpinning of the Lorenz IQO, see Foster (1985). Note that a variable population version of the Lorenz IQO could also be defined (the same is true for the minimalist IQO and the relative differentials IQO defined in Sect. 3). However, this would unnecessarily complicate the analysis since the results we would obtain in a variable population framework are essentially the same as those we obtain in the fixed population framework adopted here.

as Lorenz consistent if it follows the asymmetric part of the Lorenz IQO for comparisons between income distributions with the same mean incomes.[6]

Lorenz consistency For all $x, y \in X$, if $\mu(x) = \mu(y)$ and $x \prec_L y$, then $x P y$.

In the literature, social welfare functions are often assumed to depend on mean income and inequality only, i.e., it is assumed that there exists an inequality measure J and a function $f : (\mathbb{R}_{++} \times \mathbb{R}) \to \mathbb{R}$, increasing in the first argument and decreasing in the second, such that $W(x) = f(\mu(x), J(x))$ for all $x \in X$. In that framework, Lorenz consistency is a weak requirement for SWOs—it is sufficient that the underlying inequality measure is Lorenz consistent.[7] We note that all CES SWOs are Lorenz consistent and can be written as a function of mean income and inequality. Specifically, it can be shown that each CES SWO R_ε can be represented by a social welfare function of the form $W_\varepsilon : x \mapsto \mu(x)[1 - J_\varepsilon(x)]$, where J_ε is a Lorenz consistent inequality measure.[8] We emphasize that our results—with the sole exception of Proposition 4 in Sect. 5—do not assume that social welfare is a function of mean income and inequality only. To the contrary, we do not make any assumptions at all about the determinants of social welfare (see the discussion at the end of Sect. 3). Because the mean income-inequality representation of social welfare is popular, we will however occasionally interpret results in that light.

3 Three concepts of inequality aversion

In this section, we define three concepts for comparing degrees of inequality aversion based on Procedure ($\star$). We give a formal outline of this procedure. First, a set is determined that contains exactly all pairs of income distributions such that one income distribution is strictly more unequal than the other according to some "reference" IQO (clearly, this set is simply the asymmetric part of the reference IQO on X). These are exactly all pairs for which each SWO either implies an inequality averse choice (the less unequal income distribution is chosen), a neutral choice (indifference), or an inequality prone choice (the more unequal one is chosen)—three choices which can of course be unambiguously ranked from most inequality averse to least inequality averse. Second, two SWOs are compared with respect to the choices implied for each of the pairs of income distributions in the asymmetric part of the reference IQO: one SWO is referred to as at least as inequality averse as the other if it implies an at least as inequality averse choice for all pairs belonging to the reference set.

[6] So we use the same term for two different concepts of Lorenz consistency. However, confusion is avoided because it will always be clear from the context whether the Lorenz consistency concept for IQOs or that for SWOs is meant.

[7] Note that, for a continuous, monotonic, and separable SWO R, Lorenz consistency is satisfied if the following weaker criterion is satisfied: $\mu(x)1_n P x$ for all $x \in X$ such that x is not perfectly equal. See Chateauneuf and Moyes (2004, Proposition 1).

[8] See Atkinson (1970).

The procedure can be defined formally as follows, with $\preceq_A$ taking the role of the reference IQO.

Definition 1 *Let $\preceq_A$ be an IQO. Let R and R' be two SWOs. Then, R is said to be at least as A-inequality averse as R' if, for all $x, y \in X$ such that $x \prec_A y$, we have* (i) *if $x\,P'\,y$, then $x\,P\,y$, and,* (ii) *if $x\,I'\,y$, then $x\,R\,y$.*

As is conventional, we say that R is *more A-inequality averse than R'* if R is at least as A-inequality averse as R' while R' is not at least as A-inequality averse as R, and we say that R is *equally A-inequality averse as R'* if R is at least as A-inequality averse as R' and R' is at least as A-inequality averse as R.

In principle, any IQO can be chosen to determine the reference set $\prec_A$ in the outlined procedure. However, since different people may have different reasonable views with respect to inequality comparisons, it seems preferable to consider the common part of all these views. Now, this is exactly the role that is often attributed to the Lorenz criterion in the literature. We argue, therefore, that it is most appropriate to use as the set of pairs of income distributions for which two SWOs are compared, the set $\prec_L$. We refer to the concept of inequality aversion based on Definition 1 with $\preceq_A$ equal to $\preceq_L$ as the *L-concept*.

The L-concept is closely related to the concept of "strong risk aversion" studied by Ross (1981). Ross' concept is obtained if the L-concept is restricted to SWOs of the expected utility form, i.e., SWOs satisfying continuity, monotonicity, and separability, and if the absolute version of the Lorenz IQO is used instead of the regular (relative) version.[9]

Given the broad acceptance of the Lorenz IQO, we consider the L-concept to be the ideal concept for comparing degrees of inequality aversion, but to allow for a stronger link with the existing literature on the topic, we consider also two alternative concepts based on Definition 1 that will appear to be closer to the conventional Arrow–Pratt framework (as will be shown in Sect. 4). For these concepts, two IQOs are used that are (weaker) alternatives for the Lorenz IQO—that is, in both cases, in comparing two SWOs a set is considered which is a proper subset of $\prec_L$. The first alternative IQO we consider is the *minimalist* IQO, written as $\preceq_M$: for all $x, y \in X$,

$$x \preceq_M y \Leftrightarrow \text{there exists a scalar } e \text{ such that } x = e1_n.$$

The minimalist IQO only allows inequality comparisons between pairs of income distributions of which at least one is perfectly equal. The second alternative is the *relative differentials* IQO, written as $\preceq_{RD}$: for all $x, y \in X$,

$$x \preceq_{RD} y \Leftrightarrow \frac{x_i}{y_i} \geq \frac{x_{i+1}}{y_{i+1}} \quad \text{for all } i = 1, 2, \ldots, n-1.$$

[9] Definition 1 has, moreover, a different phrasing than the concept of Ross (1981). Statement (ii) of Proposition 1 and condition (2) in the proof of Lemma 2, are closer to the formulation used by Ross.

The relative differentials IQO, which was introduced into the literature on income distribution by Moyes (1994), says that each progressive redistribution decreases inequality. Setting $\preceq_A$ in Definition 1 equal to $\preceq_M$ or $\preceq_{RD}$, we obtain the *M-concept* and the *RD-concept*, respectively.

The M-concept is sometimes considered in the literature on risk aversion, but in a restricted version that makes the concept applicable only to SWOs of the expected utility form, i.e., SWOs that satisfy continuity, monotonicity, and separability. It is an established result in this context that, for SWOs of the expected utility form, the M-concept and the Arrow–Pratt concept are equivalent.[10] A more general result will be shown to hold in Sect. 4.

Since the three concepts of inequality aversion rely on comparisons of choices over pairs of income distributions that are members of some set that represents a view on inequality, $\prec_M$, $\prec_{RD}$, and $\prec_L$, respectively, and given the fact that $\prec_M \subset \prec_{RD} \subset \prec_L$, the following lemma is straightforwardly established. We state it without proof.

Lemma 1 *Let R and R' be two SWOs. Then, of the following three statements,* (i) *implies* (ii)*, but* (ii) *does not imply* (i)*, and* (ii) *implies* (iii)*, but* (iii) *does not imply* (ii):

(i) *R is at least as L-inequality averse as R';*
(ii) *R is at least as RD-inequality averse as R';*
(iii) *R is at least as M-inequality averse as R'.*

The relationships described in Lemma 1 also hold for the relation "is equally inequality averse as," but not for the relation "is more inequality averse than."

Lemma 1 shows that the RD-concept is more demanding than the M-concept and, in turn, the L-concept is more demanding than the RD-concept.[11] A consequence is that if, for instance, the M-concept and the L-concept yield a different conclusion, then this disagreement will typically be of the type where the M-concept ranks two SWOs whereas the L-concept does not. The converse case, as well as cases in which the M-concept and the L-concept rank two SWOs in opposite ways, are excluded by Lemma 1. In this respect, it is important to note that if two SWOs, say R and R', are incomparable according to one of the three concepts of inequality aversion, this does not simply mean that there is not sufficient evidence to refer to one SWO as at least as inequality averse as the other, but, more strongly, it means that the evidence is pointing in different directions: for some pair(s) of income distributions, R is locally more inequality

[10] See, for instance, Mas-Colell et al. (1995, Proposition 6.C.2)—the restricted version of the M-concept is close to their statement (v).

[11] It is possible also to define inequality aversion concepts that are more demanding than the L-concept—for instance, by setting $\preceq_A$ in Definition 1 equal to the IQO that extends $\preceq_L$ with Kolm's (1976) "principle of diminishing transfers," or even by setting $\preceq_A$ equal to a complete IQO corresponding to a particular Lorenz consistent inequality measure. However, since the L-concept yields impossibility results in important cases (see Theorems 2 and 4), the same is true—a fortiori—for such more demanding concepts. In other words, for the questions raised in this article, the use of these more demanding concepts would not change the implications obtained using the L-concept.

averse than R', while, for (an)other pair(s), R' is locally more inequality averse than R.

Comparisons of inequality aversion are often interpreted as comparisons of the willingness of SWOs to sacrifice mean income in return for a given decrease in inequality. Since this view of inequality aversion as essentially describing a trade-off between mean income and equality is popular, we wish to demonstrate that the L-concept, the M-concept, and the RD-concept are consistent with it—i.e., that these three concepts can be rephrased in terms of the mean income-equality trade-off. The following proposition shows that according to each of the three concepts, for all continuous and monotonic SWOs R and R', R is at least as inequality averse as R' if and only if, starting from any income distribution, R accepts a move to a given lower level of inequality at a loss of at least as much mean income as R' does.

Proposition 1 *Let $\preceq_A$ be equal to either $\preceq_M$, $\preceq_{RD}$, or $\preceq_L$. Let R and R' be two continuous and monotonic SWOs. Then, the following two statements are equivalent:*

(i) *R is at least as A-inequality averse as R';*
(ii) *for all $x, x', y \in X$ such that $x \sim_A x' \prec_A y$, if $x\,I\,y$ and $x'\,I'\,y$, then $\mu(x) \leq \mu(x')$.*

Proof Let $\preceq_A$ be equal to either $\preceq_M$, $\preceq_{RD}$, or $\preceq_L$. Let R and R' be two continuous and monotonic SWOs.

(i) $\Rightarrow$ (ii). Assume that (i) holds, i.e., R is at least as A-inequality averse as R'. Let $x, x', y \in X$ be arbitrary income distributions such that $x \sim_A x' \prec_A y$, $x\,I\,y$, and $x'\,I'\,y$. It is sufficient to show that $\mu(x) \leq \mu(x')$. Let $\lambda > 0$ be such that $\lambda x' = x$ (such a λ exists because $x \sim_A x'$). Since $x' \prec_A y$ and furthermore (i) holds, $x'\,I'\,y$ implies $x'\,R\,y$. Since also $x\,I\,y$, we have $x'\,R\,x$ by transitivity, and hence $\lambda \leq 1$ by monotonicity. We obtain that $\mu(x) \leq \mu(x')$.

(ii) $\Rightarrow$ (i). Assume that (ii) holds, i.e., for all $x, x', y \in X$ such that $x \sim_A x' \prec_A y$, $x\,I\,y$, and $x'\,I'\,y$, we have $\mu(x) \leq \mu(x')$. Let $w, z \in X$ be arbitrary income distributions such that $w \prec_A z$. It is sufficient to show that $w\,P'\,z$ implies $w\,P\,z$ and that $w\,I'\,z$ implies $w\,R\,z$. Consider first the case where $w\,P'\,z$. Let $\lambda, \lambda' > 0$ be such that $\lambda w\,I\,z$ and $\lambda' w\,I'\,z$ (such λ and λ' exist by continuity and monotonicity). Since $w\,P'\,z$, we have $w\,P'\,\lambda' w$ by transitivity, and hence $\lambda' < 1$ by monotonicity. Since (ii) holds, we have $\lambda \leq \lambda'$, and hence $\lambda < 1$. We obtain $w\,P\,z$ from $\lambda w\,I\,z$ and $\lambda < 1$ using monotonicity. The proof for the case where $w\,I'\,z$ is similar and therefore omitted. □

To conclude the section, we mention two reasons for preferring the elementary formulation used in Definition 1—i.e., the formulation in terms only of preferences over pairs of income distributions—to the more traditional formulation in terms of the mean income-equality trade-off. First, the formulation in Definition 1 has the advantage that it allows application of the inequality aversion concepts to *all* SWOs—also for instance to non-continuous SWOs, which will be useful in the discussion of extreme inequality aversion in Sect. 5. Second, a deeper concern is that an explicit reference to a mean income-equality trade-off may in certain cases misrepresent what comparisons of inequality aversion are really about. In general, there is no reason why equality should be traded off *only* with mean income. SWOs may express interest for other concerns, such as poverty alleviation for instance—then, the trade-off with mean income is just one of several trade-offs that are relevant for the idea of inequality aversion. As the neutral formulation used in Definition 1 does not refer to any *particular* trade-off, it seems to better capture the general essence of the idea of inequality aversion.

4 The three concepts and the Arrow–Pratt approach

The objective of this section is to compare the conventional Arrow–Pratt concept with the three concepts of inequality aversion that were presented in the previous section. The results for the M-concept and the RD-concept are given in Theorem 1, and the result for the L-concept is given in Theorem 2.

We first define the Arrow–Pratt concept. The analysis of Pratt (1964) concerning risk aversion has provided several equivalent concepts that can be applied to the problem of comparing degrees of inequality aversion (see also Lambert 2001, pp. 94–97). Some of these concepts can only be used to compare SWOs that can be written in the expected utility form, i.e., SWOs that satisfy continuity, monotonicity, and separability. This class is important and we shall pay attention to it in this section. However, because we wish to initially consider the entire class of continuous and monotonic SWOs, we focus on the strongest of Pratt's concepts that is applicable also to non-separable SWOs, viz., the criterion based on the equally distributed equivalent income. The *equally distributed equivalent income* for an income distribution x and an SWO R is the income, $\xi_R(x)$, that, if equally distributed, yields the same level of welfare according to R as the income distribution x.[12] Formally, for an SWO R and all $x \in X$, we have $\xi_R(x) = e$ if and only if $e1_n\, I\, x$. Note that the function ξ_R represents the SWO R. The Arrow–Pratt concept of inequality aversion is defined as follows.

Definition 2 *Let R and R' be two continuous and monotonic SWOs. Then, R is said to be at least as Arrow–Pratt inequality averse as R' if $\xi_R(x) \leq \xi_{R'}(x)$ for all $x \in X$.*[13]

The "more inequality averse than" and "equally inequality averse as" relations corresponding to the Arrow–Pratt concept are defined in the same way as for the inequality aversion concept of Definition 1.

According to the Arrow–Pratt concept, an SWO in the CES class is more inequality averse as the value of its corresponding ε is greater.[14] For this reason,

[12] See Atkinson (1970) and Kolm (1969).

[13] Note that the Arrow–Pratt concept compares, for all income distributions, how much sacrifice of mean income SWOs maximally allow in order to move from a given income distribution to a perfectly equal one—for an SWO R and an income distribution x, this sacrifice equals $[\mu(x) - \xi_R(x)]$.

[14] In fact, ε is the value of the relative Arrow–Pratt measure of risk/inequality aversion.

ε is traditionally interpreted as being a parameter of inequality aversion for the CES class. A word of caution is in order here. The parameter ε plays a double role in the CES class: besides being a parameter of inequality aversion in the Arrow–Pratt sense, it is also a parameter that measures the sensitivity of an SWO to inequality in the bottom of the income distribution relative to inequality in the top. Whereas the concepts of inequality aversion and bottom sensitivity are very different in general, they happen to coincide in the case of the CES class (see Cowell 1985).[15] When later in this section the parameter ε is discussed, it is discussed as a parameter of inequality aversion, not as a parameter of bottom sensitivity.

Although we are most interested in the L-concept for the reason specified in Sect. 3, it is convenient for expositional purposes to start with the comparison of the Arrow–Pratt concept with the M-concept and RD-concept. These concepts turn out to be closer to the Arrow–Pratt concept than the L-concept is. The following theorem summarizes the relationships between the Arrow–Pratt concept, the M-concept, and the RD-concept.

Theorem 1 *Let R and R' be two continuous and monotonic SWOs. Consider the following three statements:*

(i) *R is at least as Arrow–Pratt inequality averse as R'*;
(ii) *R is at least as M-inequality averse as R'*;
(iii) *R is at least as RD-inequality averse as R'.*

We have that

(a) *statements* (i) *and* (ii) *are equivalent*;
(b) *statement* (iii) *implies statement* (i)*, but statement* (i) *does not imply statement* (iii);
(c) *if, in addition, R and R' are separable, then statements* (i)*,* (ii)*, and* (iii) *are equivalent.*

Proof Let R and R' be two continuous and monotonic SWOs.

We first prove statement (a) of the theorem.

(i) $\Rightarrow$ (ii). Assume that (i) holds, i.e., $\xi_R(x) \leq \xi_{R'}(x)$ for all $x \in X$. Let $w, z \in X$ be arbitrary income distributions such that $w \prec_M z$. It is sufficient to show that $w\,P'\,z$ implies $w\,P\,z$ and that $w\,I'\,z$ implies $w\,I\,z$. Consider first the case where $w\,P'\,z$. We have $\xi_{R'}(w) > \xi_{R'}(z)$. Since $w \prec_M z$, we have furthermore $w = \xi_R(w)1_n = \xi_{R'}(w)1_n$. Since also $\xi_{R'}(z) \geq \xi_R(z)$ by (i), we have $\xi_R(w) > \xi_R(z)$, and we obtain $w\,P\,z$. The proof for the case where $w\,I'\,z$ is similar and therefore omitted.

(ii) $\Rightarrow$ (i). Assume that (ii) holds, i.e., R is at least as M-inequality averse as R'. Let $x \in X$ be an arbitrary income distribution. It is sufficient to show that

[15] While the concept of inequality aversion is typically related to comparisons of income distributions with different means (see the discussion of Proposition 1 in Sect. 3), the concept of bottom sensitivity is related to comparisons of income distributions with the same means (see Cowell 1985, p. 569). Unfortunately, bottom sensitivity is often also referred to as “inequality aversion” in the literature, thus blurring the significant difference in meaning between the two concepts.

$\xi_R(x) \leq \xi_{R'}(x)$. We have $\xi_{R'}(x)1_n\, I'\, x$ by definition. If $\xi_{R'}(x)1_n \sim_M x$, then we have $\xi_R(x) = \xi_{R'}(x)$. If $\xi_{R'}(x)1_n \prec_M x$, then $\xi_{R'}(x)1_n\, I'\, x$ implies $\xi_{R'}(x)1_n\, R\, x$ by (ii). Since also $\xi_R(x)1_n\, I\, x$ by definition, we have $\xi_{R'}(x)1_n\, R\, \xi_R(x)1_n$ by transitivity. We obtain $\xi_R(x) \leq \xi_{R'}(x)$ by monotonicity.

Next, we prove statement (b) of the theorem.

(iii) $\Rightarrow$ (i). This follows from statement (a) of the theorem and Lemma 1.

(i) $\not\Rightarrow$ (iii). We consider an example where R is at least as Arrow–Pratt inequality averse as R', but R is not at least as RD-inequality averse as R'. Let the equally distributed equivalent incomes (i.e., social welfare functions) for R and R' be given by

$$\xi_R : x \longmapsto \frac{7x_1 + 2x_2 + w_3x_3 + w_4x_4 + \cdots + w_{n-1}x_{n-1} + x_n}{10 + w_3 + w_4 + \cdots + w_{n-1}},$$

and

$$\xi_{R'} : x \longmapsto \frac{5x_1 + 4x_2 + w_3x_3 + w_4x_4 + \cdots + w_{n-1}x_{n-1} + x_n}{10 + w_3 + w_4 + \cdots + w_{n-1}},$$

respectively, where $w_3, w_4, \ldots, w_{n-1}$ are positive scalars (the same for R and R'). We have $\xi_R(x) \leq \xi_{R'}(x)$ for all $x \in X$, and hence R is at least as Arrow–Pratt inequality averse as R'. Now, let $x, y \in X$ be such that $(x_1, x_2, x_n) = (15, 250, 260)$, $(y_1, y_2, y_n) = (10, 200, 400)$, and $x_i = y_i = 255$ for all $i = 3, 4, \ldots, n-1$. We have $x \prec_{RD} y$, $y\, P\, x$, and $x\, P'\, y$, and hence R is not at least as RD-inequality averse as R'.

The proof of statement (c) of the theorem is given in an appendix. □

We mentioned in the previous section that the M-concept and the Arrow–Pratt concept are equivalent for continuous, monotonic, and separable SWOs. As statement (a) of Theorem 1 shows, this equivalence also holds without separability. This result is not very surprising, given the fact that the definitions of both the M-concept and the Arrow–Pratt concept refer to preferences over pairs of income distributions of which one is perfectly equal. Statement (b) shows that, if we take the step from the minimalist IQO to the relative differentials IQO as the underlying inequality criterion for the concept of inequality aversion, then we move away from convention. The inconsistency of the RD-concept and the Arrow–Pratt concept consists of there being SWOs such that the Arrow–Pratt concept ranks them while the RD-concept does not. Finally, statement (c) shows that the RD-concept and the Arrow–Pratt concept agree on how to rank any pair of SWOs of the expected utility form.

With respect to the M-concept and the RD-concept we may conclude that the former, and to a lesser extent the latter, support the claims made traditionally in the literature on the basis of the Arrow–Pratt concept. An important question we now turn to is whether the favoured L-concept is consistent with these claims. We already know, by Lemma 1 and statement (b) of Theorem 1, that the L-concept and the Arrow–Pratt concept cannot be equivalent for the entire class of monotonic and continuous SWOs, so the question becomes whether this equivalence holds for the expected utility class of SWOs (as for the

RD-concept), or at least for the popular CES subclass. This appears *not* to be the case. There are several pairs of CES SWOs R_ε and $R_{\varepsilon'}$ such that $\varepsilon > \varepsilon'$, and several pairs of income distributions $x, y \in X$ such that $x \prec_L y$, for which we have $y \, P_\varepsilon \, x$ while $x \, P_{\varepsilon'} \, y$. This is illustrated in the following example.

Example 1 The example is for the case $n = 3$. Take the income distributions $x = (19, 57, 76)$ and $y = (20, 20, 130)$. We have $x \prec_L y$. However, for all CES SWOs with ε such that $0.403 < \varepsilon < 14.513$, we have $x \, P_\varepsilon \, y$, while for all CES SWOs with $\varepsilon > 14.514$, we have $y \, P_\varepsilon \, x$.[16]

Using a result by Ross (1981) it is possible to draw even stronger conclusions with respect to the CES class. Ross' critique of the Arrow–Pratt framework can be interpreted as a confrontation of the L-concept and the M-concept in the framework of expected utility theory. The following lemma is based on one of his results.

Lemma 2 *Let R_u and R_v be two continuous, monotonic, and separable SWOs such that the respective corresponding utility functions, u and v, are three times differentiable and concave. Then, the following two statements are equivalent*:

(i) *R_u is at least as L-inequality averse as R_v*;
(ii) *there exist a non-increasing and concave function $f : \mathbb{R}_{++} \to \mathbb{R}$ and a scalar $\lambda > 0$ such that $u(t) = \lambda v(t) + f(t)$ for all $t \in \mathbb{R}_{++}$.*

Proof Let R_u and R_v be two continuous, monotonic, and separable SWOs such that the respective corresponding utility functions, u and v, are three times differentiable and concave.

Ross (1981, Theorem 3) shows that (ii) is equivalent to the following condition: for all $x, y \in X$ such that $\mu(x) = \mu(y)$ and $x \prec_L y$, if $(x - \pi_u 1_n) \, I_u \, y$ and $(x - \pi_v 1_n) \, I_v \, y$, then $\pi_u \geq \pi_v$. Inspection of Ross' proof reveals that (ii) is also equivalent to the following similar condition:

$$\text{for all } x, y \in X \text{ such that } \mu(x) = \mu(y) \text{ and } x \prec_L y, \quad \text{if } \gamma_u x \, I_u \, y \text{ and } \gamma_v x \, I_v \, y, \text{ then } \gamma_u \leq \gamma_v. \tag{2}$$

What remains to be shown is that the condition in (2) is equivalent to (i). First we show that (2) is equivalent to

$$\text{for all } x, x', y \in X \text{ such that } x \sim_L x' \prec_L y, \text{ if } x \, I_u \, y \text{ and } x' \, I_v \, y, \text{ then } \mu(x) \leq \mu(x'). \tag{3}$$

It is immediate that (3) implies (2). That (2) implies (3) follows from the fact that if there exist $x, x', y \in X$ such that $x \sim_L x' \prec_L y$, $x \, I_u \, y$ and $x' \, I_v \, y$, then

[16] Note that we have $\mu(x) < \mu(y)$ in the example. This is no coincidence since if we would have $\mu(x) \geq \mu(y)$ and $x \prec_L y$, then all CES SWOs would strictly prefer x over y, as can be easily established using the fact that all these SWOs satisfy Lorenz consistency.

there exists a $z \in X$ such that $\mu(y) = \mu(z)$ and scalars γ_u, γ_v such that $x = \gamma_u z$ and $x' = \gamma_v z$. Now, since (3) is equivalent to (i) by Proposition 1, the required result follows. □

It can be shown now that in the entire class of CES SWOs there are no two SWOs that can be compared using the L-concept of inequality aversion.

Theorem 2 *Let R_ε and $R_{\varepsilon'}$ be two CES SWOs such that $\varepsilon \neq \varepsilon'$. Then, R_ε and $R_{\varepsilon'}$ are incomparable according to the L-concept, i.e., R_ε is not at least as L-inequality averse as $R_{\varepsilon'}$, and $R_{\varepsilon'}$ is not at least as L-inequality averse as R_ε.*

Proof Let R_ε and $R_{\varepsilon'}$ be CES SWOs such that $\varepsilon \neq \varepsilon'$. Without loss of generality, let $\varepsilon > \varepsilon'$.

Since $\varepsilon > \varepsilon'$, R_ε is more M-inequality averse than $R_{\varepsilon'}$. Hence, $R_{\varepsilon'}$ is not at least as L-inequality averse as R_ε by Lemma 1. What remains to be shown is that R_ε is not at least as L-inequality averse as $R_{\varepsilon'}$. Seeking a contradiction, assume that R_ε is at least as L-inequality averse as $R_{\varepsilon'}$. Then, by Lemma 2, there exist a non-increasing and concave function $f : \mathbb{R}_{++} \to \mathbb{R}$ and a scalar $\lambda > 0$ such that

$$\frac{t^{1-\varepsilon}}{1-\varepsilon} = \lambda \frac{t^{1-\varepsilon'}}{1-\varepsilon'} + f(t) \quad \text{for all } t \in \mathbb{R}_{++}.$$

Non-increasingness and concavity of f imply

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = t^{-\varepsilon} - \lambda t^{-\varepsilon'} \leq 0 \quad \text{for all } t \in \mathbb{R}_{++}, \tag{4}$$

and

$$\frac{\mathrm{d}f^2(t)}{\mathrm{d}t^2} = -\varepsilon t^{-(1+\varepsilon)} + \lambda \varepsilon' t^{-(1+\varepsilon')} \leq 0 \quad \text{for all } t \in \mathbb{R}_{++}. \tag{5}$$

From (4) and (5) it follows that

$$\lambda \geq t^{-(\varepsilon-\varepsilon')} \quad \text{for all } t \in \mathbb{R}_{++}, \tag{6}$$

and

$$\lambda \leq \frac{\varepsilon}{\varepsilon'} t^{-(\varepsilon-\varepsilon')} \quad \text{for all } t \in \mathbb{R}_{++}, \tag{7}$$

respectively. Since the functions $t \mapsto t^{-(\varepsilon-\varepsilon')}$ and $t \mapsto (\varepsilon/\varepsilon')t^{-(\varepsilon-\varepsilon')}$ map $\mathbb{R}_{++}$ *onto* $\mathbb{R}_{++}$, there exist $s, t \in \mathbb{R}_{++}$ such that $s^{-(\varepsilon-\varepsilon')} > (\varepsilon/\varepsilon')t^{-(\varepsilon-\varepsilon')}$. By consequence, λ cannot satisfy both (6) and (7) and we have a contradiction. □

The CES class of SWOs is often considered to be very useful in practice because, according to the conventional Arrow–Pratt approach, it encompasses a continuum of positions with respect to inequality aversion from the completely non-egalitarian mean income rule ($\varepsilon = 0$) to leximin ($\varepsilon \to \infty$), which is often viewed as constituting the extreme case of inequality aversion (see Sect. 5). The class owes its popularity furthermore to the fact that it has attractive properties from the theoretical perspective: all CES SWOs satisfy the basic axioms continuity, monotonicity, and separability, and allow a natural decomposition into mean income and a Lorenz consistent inequality measure as explained in Sect. 2. However, the deep inconsistency between, on the one hand, the conventional interpretation of the parameter ε and, on the other hand, the L-concept may be seen as somewhat damaging for the CES class to operate as a canonical class of SWOs. The problem is aggravated by the fact that all members of the CES class ascribe importance to the Lorenz IQO—and thus the L-concept—because they are all Lorenz consistent. Is it possible to find another class of SWOs which both has attractive properties and encompasses a continuum of degrees of inequality aversion according to the L-concept? Although we shall not attempt to answer this question here, we wish to note that a sacrifice will have to be made irrespective of the direction in which an answer is sought. For instance, the analysis of Ross (1981) can be used to construct a class of SWOs to play a role similar to that of the CES class, in which case continuity, monotonicity, and separability will still be satisfied. However, it may possibly be seen as a drawback that in that case the natural link between welfare and an underlying criterion of (Lorenz consistent) inequality will be lost. Alternatively, such a natural link can be taken as a starting point to construct an alternative to the CES class, but at the cost of separability. [17]

5 Extreme inequality aversion

In this section, we characterize, for each of the three concepts proposed in Sect. 3, the classes of SWOs that reconcile monotonicity with an extreme form of inequality aversion.[18] The results for the M-concept and the RD-concept are given in Theorem 3, and the result for the L-concept is given in Theorem 4.

Conventionally, maximin and leximin are seen as typical examples of extremely inequality averse SWOs. Both SWOs give absolute priority to the worst off. Maximin furthermore implies indifference in all cases in which the worst off is equally well off, i.e., an SWO R is called *maximin* if, for all $x, y \in X$, we have $x R y \Leftrightarrow x_1 \geq y_1$. Leximin, on the other hand, gives priority to the second worst off in the cases where the worst off is equally well off in both alternatives, and so on, i.e., an SWO R is called *leximin* if, for all $x, y \in X$, we have

$$x R y \quad \Leftrightarrow \text{either } x = y\text{, or, there is an integer } k \text{ such that}$$
$$x_i = y_i \text{ for all } i < k\text{, and } x_k > y_k.$$

Maximin and leximin are both members of the class of weakly maximin SWOs, which is the class of SWOs that have in common the asymmetric part of maximin, i.e., an SWO R is called *weakly maximin* if, for all $x, y \in X$, we have that if $x_1 > y_1$, then $x P y$. It can be shown that maximin is the only continuous

[17] See Champernowne and Cowell (1998, pp. 107–108) on a similar point.

[18] How the ideals of extreme inequality aversion and monotonicity can be combined is an important question in egalitarian social ethics. See Tungodden (2003, pp. 10–23) for an overview of the economic and philosophical literature concerning this topic.

member of the class of weakly maximin SWOs and that leximin is the only separable member of the class. It will be of interest to see what role leximin plays in our analysis, since it is the only popular SWO that is commonly viewed as combining extreme inequality aversion with monotonicity—maximin, by contrast, does not satisfy monotonicity.

The starting point of our analysis is the following definition of the idea of extreme inequality aversion.

Definition 3 *Let $\mathscr{S}$ be a class of SWOs. An SWO R is said to be extremely inequality averse in $\mathscr{S}$ if R is a member of $\mathscr{S}$ and R is at least as inequality averse as each member of $\mathscr{S}$.*

This definition assures that an extremely inequality averse SWO in $\mathscr{S}$ never implies a choice over a pair of income distributions that is less inequality averse than that implied by any other member of $\mathscr{S}$. Note also that all extremely inequality averse SWOs are equally inequality averse.

In what follows, we identify the members of the class of monotonic SWOs that are extremely inequality averse according to the M-concept, the RD-concept, and the L-concept. Since we do not require continuity, the standard Arrow–Pratt concept cannot be applied in this context—however, it is natural to interpret the M-concept as being the evident extension of the Arrow–Pratt concept capable of such comparisons. Again, it is convenient to begin the analysis by considering the M-concept and the RD-concept.

Theorem 3 *Let R be a monotonic SWO. Then, the following five statements are equivalent:*

(i) *R is extremely M-inequality averse in the class of monotonic SWOs*;
(ii) *R is extremely RD-inequality averse in the class of monotonic SWOs*;
(iii) *for all $x, y \in X$ such that $x \not< y$, we have that if $x \prec_M y$, then $x P y$*;
(iv) *for all $x, y \in X$ such that $x \not< y$, we have that if $x \prec_{RD} y$, then $x P y$*;
(v) *R is weakly maximin.*

Proof Let R be a monotonic SWO.

(i) $\Leftrightarrow$ (iii). First we show that (i) implies (iii). Seeking a contradiction, assume that (i) holds while (iii) does not. Let $x, y \in X$ be arbitrary income distributions such that $x \not< y$, $x \prec_M y$, and $y R x$ (such x and y exist since (iii) does not hold). Let R' be an arbitrary monotonic SWO such that $x P' y$ (such an R' exists since $x \not< y$). Since $x \prec_M y$, $y R x$, and $x P' y$, we have that R is not at least as M-inequality averse as R', which contradicts (i). That (iii) implies (i) is immediate.

(iii) $\Leftrightarrow$ (v). First we show that (iii) implies (v). Seeking a contradiction, assume that (iii) holds while (v) does not. Let $x, y \in X$ be arbitrary income distributions such that $x_1 > y_1$ and $y R x$ (such x and y exist since R is not weakly maximin). Reflexivity and monotonicity imply that $x R x_1 1_n$. Since also $y R x$, we have $y R x_1 1_n$ by transitivity. Since $x_1 > y_1$, we have $x_1 1_n \not< y$ and since R is monotonic, $y R x_1 1_n$ implies $x_1 1_n \prec_M y$. We have $y R x_1 1_n$ while $x_1 1_n \not< y$ and $x_1 1_n \prec_M y$, which contradicts (iii). Second we show that (v) implies (iii).

Assume that (v) holds. Let $x, y \in X$ be arbitrary income distributions such that $x \prec_M y$ and $x \not< y$. It is sufficient to show that xPy. Now, $x \prec_M y$ implies $x = x_1 1_n$, and $x_1 1_n \not< y$ implies $x_1 > y_1$. Since R is weakly maximin, we have xPy.

(ii) $\Leftrightarrow$ (iv). The proof is similar to that of the equivalence of (i) and (iii) and is therefore omitted.

(iv) $\Leftrightarrow$ (v). First we show that (iv) implies (v). Since $\prec_M \subset \prec_{RD}$, (iv) implies (iii). Since also (iii) implies (v), the required result follows. Second we show that (v) implies (iv). Assume that (v) holds. Let $x, y \in X$ be arbitrary income distributions such that $x \not< y$ and $x \prec_{RD} y$. It is sufficient to show that xPy. Because $x \not< y$, there exists an $i \in N$ such that $x_i/y_i > 1$. Since furthermore $x \prec_{RD} y$, we have $x_1/y_1 > 1$. Hence, we have xPy since R is weakly maximin. □

The equivalence of (i) and (v) in Theorem 3 says that the case of extreme M-inequality aversion in the class of monotonic SWOs is covered by the monotonic weakly maximin SWOs. To a certain extent, this result supports the conventional view that leximin constitutes the case of extreme inequality aversion. The reason is that the literature focuses virtually exclusively on separable SWOs when studying extreme inequality aversion, combined with the fact that leximin is the only separable weakly maximin SWO.[19] The finding that (i) and (v) are equivalent is important for two reasons. Firstly, given Lemma 1, it follows from this result that the classes of extremely inequality averse SWOs that are implied by the RD-concept and the L-concept must be subsets of the class of monotonic weakly maximin SWOs. Secondly, it presents another way of seeing why the M-concept is unattractive. As an illustration of this point, consider the following SWO R: for all $x, y \in X$, we have that

$$\text{if } x_1 > y_1, \text{ then } xPy, \quad \text{and} \quad \text{if } x_1 = y_1, \text{ then } xRy \Leftrightarrow \mu(x) \geq \mu(y).$$

This SWO is both monotonic and weakly maximin. Now note that whenever two income distributions have the same lowest incomes, this SWO ranks them according to the completely non-egalitarian mean income rule.[20] Probably, many would hesitate to refer to such an SWO as extremely inequality averse, thus implicitly accepting that the M-concept is too undemanding as a criterion for comparing degrees of inequality aversion. However, as the equivalence of (ii) and (v) shows, moving on to the more demanding RD-concept does not

[19] The interpretation of leximin as being extremely inequality averse can be defended on the basis of the Arrow–Pratt concept. Hammond (1975) has demonstrated that leximin can be interpreted as the limit case, $\varepsilon \to \infty$, of the CES class of SWOs, a point which Lambert (2001, Theorem 4.4) has generalized with respect to the entire class of continuous, monotonic, and separable SWOs. In Bosmans (2007), it is shown that using an approach analogous to that of Hammond and Lambert, the weakly maximin class can be identified as extremely inequality averse on the basis of the Arrow–Pratt concept if separability is dropped. Theorem 3 confirms the latter result.

[20] Note that the comparison of such income distributions is probably even quite common in practice—think of a change in the tax system that leaves the existing minimally guaranteed income unaffected.

solve anything: the class of monotonic weakly maximin SWOs is still identified as the extremely inequality averse subclass of the class of monotonic SWOs. Before we consider which monotonic weakly maximin SWOs survive the test of Definition 3 when we move to the L-concept, we consider the other statements of Theorem 3.

The conditions expressed in statements (iii) and (iv) of Theorem 3 constitute a natural way of giving meaning to extreme inequality aversion for SWOs that satisfy monotonicity—the conditions say that one should prefer, for any pair of income distributions, the one which is less unequal (according to the minimalist IQO and the relative differentials IQO in statements (iii) and (iv), respectively) unless the income distribution is worse for some and better for none. In a recent study on the possibility of combining extreme inequality aversion and monotonicity, Tungodden and Vallentyne (2005) have taken natural conditions as those expressed in statements (iii) and (iv) as a starting point (so, relying only implicitly on the concepts defined in our Definitions 1 and 3). They have considered a condition similar to that of statement (iii) and also show that statements (iii) and (v) are equivalent. Later, we draw a more interesting parallel between the present work and theirs.

Now, we come to the important question of which SWOs are extremely inequality averse according to the L-concept. Note first that while the M-concept and the RD-concept identify all weakly maximin SWOs as extremely inequality averse, according to the L-concept *no* member of this class is extremely inequality averse.

Proposition 2 *Let R be a continuous and monotonic SWO satisfying Lorenz consistency. Then, there exist $x, y \in X$ for which $x \prec_L y$, such that we have $x P y$ while all weakly maximin SWOs strictly prefer y to x.*

Proof Let R be a continuous and monotonic SWO satisfying Lorenz consistency. Let $y \in X$ be an arbitrary income distribution such that $y_1 \leq y_2 < y_3$. Let $x^\lambda = (\lambda y_1, \lambda t, \lambda t, \ldots, \lambda t)$ be an n-dimensional vector with $t = \sum_{i=2}^{n} y_i/(n-1)$ and λ a positive scalar. Note that $x^\lambda \in X$ and $x^\lambda \prec_L y$ for all allowed values of λ. If $\lambda = 1$, then $x^\lambda P y$ by Lorenz consistency. Hence, continuity and monotonicity imply that there exist values of λ such that $0 < \lambda < 1$ and $x^\lambda P y$. Now, for all such values of λ, we have $\lambda y_1 < y_1$ and hence that y is strictly preferred to x^λ by all weakly maximin SWOs. □

Proposition 2 demonstrates that, given the L-concept, the weakly maximin SWOs do not only fail the test of extreme inequality aversion described in Definition 3, they do so in a particularly bad way. The proposition says that, for instance, it is possible to find a pair of income distributions such that a CES SWO with ε arbitrarily close to, but greater than, zero, and hence arbitrarily close to the completely non-egalitarian mean income rule, is locally more inequality averse than all weakly maximin SWOs for this pair. Because each extremely L-inequality averse SWO in the class of monotonic SWOs must be weakly maximin by Lemma 1 and Theorem 3, the following result follows immediately from Proposition 2.

Theorem 4 *There is no SWO that is extremely L-inequality averse in the class of monotonic SWOs.*

So, we conclude that if we accept the L-concept, then extreme inequality aversion is incompatible with monotonicity. In their work, Tungodden and Vallentyne (2005) reach a similar conclusion. However, they implicitly use a criterion that lies in between the M-concept and the RD-concept, and find an incompatibility.[21] This is possible because they use a slightly (but significantly) different framework than the one used here: their result is driven by the fact that they reject anonymity as a property of SWOs, but accept it for IQOs. The present study shows that without this assumption, there is no incompatibility between their version of extreme inequality aversion and monotonicity (this is implied by the equivalence of (ii) and (v) in Theorem 3), but that the incompatibility crops up again when the L-concept is accepted (Theorem 4).[22]

What should egalitarians who agree with the L-concept and want both monotonicity and extreme inequality aversion choose as an SWO? It might at first glance seem natural to regard leximin or other monotonic weakly maximin SWOs as being "close enough"—these SWOs satisfy a necessary condition for being extremely inequality averse (they are extremely inequality averse if one looks only at the pairs in $\prec_M$ or $\prec_{RD}$), and a sufficient condition cannot be satisfied (being extremely inequality averse for those in $\prec_L$ is impossible), hence why not content ourselves with these? Proposition 2 illustrates already how unattractive it is to settle for a conclusion based on the less demanding M-concept and RD-concept if the L-concept is the one which is deemed ideal. There is also a deeper reason for extreme egalitarians not to (necessarily) focus on the class of weakly maximin SWOs. It is perfectly acceptable to consider the pairs ordered by the minimalist IQO (i.e., the set $\prec_M$) as not being more important than some alternative set of pairs ordered by the Lorenz IQO (i.e., a subset of $\prec_L$ which differs from $\prec_M$). If one accepts the Lorenz IQO, these former pairs of income distributions are not special in any way. If such an alternative set of pairs is used in a criterion for comparing degrees of inequality aversion, in accordance with the explanation at the beginning of Sect. 3, then the set of extremely inequality averse monotonic SWOs need not be empty, nor contain any weakly maximin SWOs. For instance, if the income distributions from Example 1 are members of this alternative set, then none of the weakly maximin SWOs pass the test of extreme inequality aversion of Definition 3, while (depending on the other elements of the set) other SWOs may pass the test.

To conclude the section, we consider two alternative ways of giving meaning to the view that inequality reduction should always be preferred unless no one

21 More precisely, they use a condition similar to that stated in statements (iii) and (iv) of Theorem 3, but with, instead of the minimalist or relative differentials IQO, an IQO that is a proper subrelation of the relative differentials IQO and a proper superrelation of the minimalist IQO.

22 In Tungodden (2000) it is also shown that, without rejecting anonymity, their extreme inequality aversion condition and monotonicity are compatible.

gains by it. However, as we shall see, neither alternative produces a convincing way out of the incompatibility.

The first alternative is to consider the SWOs for which no monotonic SWO is more inequality averse according to the L-concept, instead of the ones that are at least as inequality averse as all the other monotonic ones according to the L-concept (as in Definition 3). Consider the following definition of this alternative concept of "maximal inequality aversion."

Definition 4 *Let $\mathscr{S}$ be a class of SWOs. An SWO R is said to be maximally inequality averse in $\mathscr{S}$ if R is a member of $\mathscr{S}$ and no member of $\mathscr{S}$ is more inequality averse than R.*[23]

The subset of maximally L-inequality averse SWOs in the class of monotonic SWOs is not empty: as the following proposition shows, at least leximin is a member.

Proposition 3 *Leximin is maximally L-inequality averse in the class of monotonic SWOs.*

Proof It is sufficient to show that there does not exist a monotonic SWO R that is more L-inequality averse than leximin. Seeking a contradiction, assume that such an R does exist. Let $x, y \in X$ be such that $x \prec_L y$, $x\,R\,y$, and leximin strictly prefers y to x. Since leximin strictly prefers y to x, there exists a $k > 1$ such that $x_i = y_i$ for all $i = 1, 2, \ldots, k-1$, and $x_k < y_k$. Let $z \in X$ be such that $z_i = x_i = y_i$ for all $i = 1, 2, \ldots, k-1$, $x_k < z_k < \min\{\sum_{i=k}^{n} x_i/(n-k+1), y_k\}$, and $z_i = z_k$ for all $i = k+1, k+2, \ldots, n$. Monotonicity implies $y\,P\,z$. Since also $x\,R\,y$, we have $x\,P\,z$ by transitivity. Now, $z \prec_L x$ and leximin strictly prefers z to x. By consequence, R is not more L-inequality averse than leximin and we have a contradiction. □

However, the concept of maximal inequality aversion seems too undemanding, because it is not excluded that there are SWOs, which are themselves unlikely candidates for being considered extremely inequality averse, that are more inequality averse for at least some pairs of income distributions—in the case of leximin, Proposition 2 should suffice to make this point.

A second alternative is to start from the view that SWOs are functions of an underlying inequality measure or IQO, a view not uncommon in the literature as we saw in Sect. 2. In that perspective, the following approach to combining monotonicity and an absolute preference for inequality reduction seems reasonable: choose an SWO that, for all pairs of income distributions to which monotonicity applies, follows monotonicity, and, for all pairs to which monotonicity does not apply, prefers the income distribution which minimizes inequality according to some IQO (or some inequality measure). Note that this approach does not require a complete IQO since the IQO need not order

[23] The distinction between extreme inequality aversion and maximal inequality aversion is analogous to the distinction made by Sen (1997) between optimization and maximization in individual choice theory.

pairs of income distributions to which monotonicity applies—it does have to order all pairs to which monotonicity does not apply, however. The question is whether it is possible to find an SWO and a corresponding IQO that satisfy the required condition. First we need to consider some minimal criteria that a "sensible" IQO ought to satisfy. The first is that it should have the minimalist IQO as a subrelation. The second is that it satisfies some invariance criterion. An invariance criterion defines the transformation that if applied to all incomes leaves inequality invariant. For instance, the invariance criterion underlying the Lorenz IQO and the relative differentials IQO is scale invariance, which says: for all $x \in X$ and all scalars $\lambda > 0$, we have $x \sim \lambda x$. However, we will demand only that a much weaker invariance criterion is satisfied. Minimal invariance says that for any given income distribution there must exist an income distribution in which everyone is better off and which is at least as unequal as the given income distribution.

Minimal invariance For all $x \in X$, there is an $x' \in X$ such that $x' \gg x$ and $x \preceq x'$.

The following proposition shows that no SWO and IQO with the described properties exist.

Proposition 4 *Let R be a monotonic SWO. Let* $\preceq$ *be an IQO that satisfies minimal invariance and for which* $\prec \supset \prec_M$*. Then, the following condition is not satisfied: for all* $x, y \in X$ *such that* $x \not\prec y$*, we have* $x \prec y \Leftrightarrow x P y$*.*

Proof Let R be a monotonic SWO. Let $\preceq$ be an IQO that satisfies minimal invariance and for which $\prec \supset \prec_M$. Seeking a contradiction, assume that the condition stated in the proposition is satisfied. The condition implies statement (i) of Theorem 3 since $\prec \supset \prec_M$. Hence, R is weakly maximin by Theorem 3. Let $x \in X$ be an arbitrary income distribution such that $x_1 < x_2 < x_3$. Let $x' \in X$ be an arbitrary income distribution such that $x' \gg x$ and $x \preceq x'$ (such an x' exists by minimal invariance). Now consider a $y \in X$ such that $x_1 < y_1 < x'_1$, $y_2 < x_2 < x'_2$, and $x_3 < x'_3 < y_3$. Since R is weakly maximin, we have $y P x$ and $x' P y$. By the condition stated in the proposition, $y \not\prec x$ and $y P x$ imply $y \prec x$, and $x' \not\prec y$ and $x' P y$ imply $x' \prec y$. Since $x' \prec y$ and $y \prec x$, we have $x' \prec x$ by transitivity, which contradicts $x \preceq x'$. □

Proposition 1 says that, no matter which ("sensible") IQO is considered, the following is true: each monotonic SWO (so also leximin and each of the other monotonic weakly maximin SWOs) will for some pairs of income distributions choose the income distribution that is not the least unequal of the two according to the given IQO, and this without this choice being directly imposed on the SWO by monotonicity.

6 Concluding remarks

In this article, we studied a straightforward dominance procedure for comparing SWOs with respect to degree of inequality aversion. We considered three versions of the procedure based on three inequality criteria: (i) the L-concept, which we argued to be the ideal version, (ii) the M-concept, which is roughly equivalent to the traditional Arrow–Pratt approach, and (iii) the RD-concept, which is intermediate in strength between the other two concepts.

It was shown that the L-concept is in general incompatible with the M-concept. In the case of the CES class of SWOs, the difference between the conclusions produced by the two concepts was especially pronounced: whereas the M-concept ranks all members of this class, the L-concept ranks none. As we have said already, it would be interesting to think about theoretically agreeable alternatives to the CES class of which the members can be ranked using the L-concept and which covers a wide spectrum of positions with respect to inequality aversion. Probably, the most attractive solution is to give up separability and to consider classes of SWOs such as, for instance, that given by $W_\alpha : x \mapsto \mu(x)[1 - J(x)]^\alpha$, where J is a Lorenz consistent inequality measure: here α is a parameter that measures inequality aversion in accordance with the L-concept. It may be interesting to see whether classes of SWOs in the spirit of this example can be constructed in a theoretically and philosophically sound way starting directly from the idea of the natural decomposition of welfare in mean income and inequality.

We showed, furthermore, that if we accept the L-concept, then monotonicity and extreme inequality aversion are incompatible. Hence, egalitarians committed to monotonicity have to content themselves with being less than extremely inequality averse: it is always possible to find pairs of income distributions for which a less inequality averse choice than possible must be made. Those who are attracted to both the ideals of monotonicity and extreme inequality aversion have to determine which of the two to weaken. We have discussed that if extreme inequality aversion is weakened, nothing forces one to opt for a weakly maximin SWO such as leximin. It is perfectly possible to choose a different set over which one wants to make inequality averse choices than the set that forces one to give full priority to the worst off. The other possibility, not yet discussed, is to weaken monotonicity. For instance, a possibility is to demand only *ray-monotonicity*: for all $x \in X$ and all $\lambda > 1$, we have $\lambda x P x$. It can easily be shown that there exist SWOs that are extremely inequality averse according to the L-concept in the class of ray-monotonic SWOs.[24] Interestingly, not only does the weakening to ray-monotonicity make it possible to have extremely inequality averse SWOs, but none of them is weakly maximin (and this is true even if we use the M-concept instead of the L-concept). In other words, whichever of the two ideals egalitarians choose to weaken in order to deal with the incompatibility, they should not feel required to restrict their consideration to leximin or other weakly maximin SWOs.

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[24] Consider the example of an SWO R: for all $x, y \in X$, we have that if $x \prec y$, then $x P y$, and if $x \sim y$, then $[x R y \Leftrightarrow \mu(x) \geq \mu(y)]$, where $\preceq$ is a Lorenz consistent and complete IQO.

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Appendix

Proof of Theorem 1, statement (c) Let R_u and R_v be two continuous, monotonic, and separable SWOs with u and v as the respective corresponding utility functions. Let R_u be at least as Arrow–Pratt inequality averse as R_v. Given statements (a) and (b) of Theorem 1, it is sufficient to show that R_u is at least as RD-inequality averse as R_v. Let $x, y \in X$ be arbitrary income distributions such that $x \prec_{RD} y$. We have to show that $x\, P_v\, y$ implies $x\, P_u\, y$ and that $x\, I_v\, y$ implies $x\, R_u\, y$. If $x > y$ or $y > x$, then the required conclusion follows immediately from monotonicity. Therefore, we assume that neither $x > y$ nor $y > x$ holds.

We now construct an income distribution y', which differs from x in at most one component and which is such that $y'\, I_v\, y$. We use the following algorithm to transform y into y'. In each step except the last, at least one component not equal to the corresponding one in x is replaced by the component from x. Hence, there is a finite number of steps, say m. Let $k \geq 1$ and $\ell \leq n$ be the elements of N such that $x_i > y_i$ for all $i = 1, 2, \ldots, k$, $x_i = y_i$ for all $i = k+1, k+2, \ldots, \ell - 1$, and $x_i < y_i$ for all $i = \ell, \ell+1, \ldots, n$. Such k and ℓ exist because $x \prec_{RD} y$, while neither $x > y$, nor $y > x$. Moreover, we have $k < \ell$.

Step $s = 1, 2, \ldots, m$: Let $y^0 = y$. Let y^{s-1} be the income distribution resulting from the previous step if $s = 2, 3, \ldots, m$. If y^{s-1} differs from x in at most one component, then the algorithm ends and $y' = y^{s-1}$. Else, income distribution y^s is constructed from y^{s-1} by replacing the components corresponding to arbitrary $i, j \in N$ such that $i \leq k < \ell \leq j$. Note that we have $y_i^{s-1} < x_i < x_j < y_j^{s-1}$. The vector y^s is constructed as follows:

- if $v(x_i) + v(x_j) = v(y_i^{s-1}) + v(y_j^{s-1})$, then y^s is equal to y^{s-1} with the ith component replaced by x_i and the jth component replaced by x_j;
- if $v(x_i) + v(x_j) > v(y_i^{s-1}) + v(y_j^{s-1})$, then y^s is equal to y^{s-1} with the ith component replaced by t and the jth component replaced by x_j, where t is such that $y_i^{s-1} < t < x_i$ and $v(x_i) + v(x_j) = v(t) + v(y_j^{s-1})$ (t exists by continuity and monotonicity);
- if $v(x_i) + v(x_j) < v(y_i^{s-1}) + v(y_j^{s-1})$, then y^s is equal to y^{s-1} with the ith component replaced by x_i and the jth component replaced by t, where t is such that $x_n < t < y_j^{s-1}$ and $v(x_1) + v(x_n) = v(y_i^{s-1}) + v(t)$ (t exists by continuity and monotonicity).

Recall that we have to show that $x\, P_v\, y$ implies $x\, P_u\, y$ and that $x\, I_v\, y$ implies $x\, R_u\, y$. Because the vector y' constructed above differs in only one component from x, and because of monotonicity and reflexivity, there are only three possible cases: both $x\, P_v\, y'$ and $x\, P_u\, y'$, both $x\, I_v\, y'$ and $x\, I_u\, y'$, or both $y'\, P_v\, x$

and $y' P_u x$. Hence, given transitivity, it is sufficient to show that $y' I_v y$ implies $y' R_u y$. It is known from Pratt (1964) that if R_u is at least as Arrow–Pratt inequality averse as R_v, then $u = f \circ v$ where the function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and concave. Consider an arbitrary step $s = 1, 2, \ldots, m-1$ in the algorithm given above. This step transforms y^{s-1} into y^s by changing two of its components, say y_i^{s-1} and y_j^{s-1} with $i < j$. Since no other components change, $y^{s-1} I_v y^s$ is equivalent to $v(y_i^{s-1}) + v(y_j^{s-1}) = v(y_i^s) + v(y_j^s)$ or $v(y_j^{s-1}) - v(y_j^s) = v(y_i^s) - v(y_i^{s-1})$. Since, furthermore, $v(y_j^{s-1}) > v(y_j^s) > v(y_i^s) > v(y_i^{s-1})$, we have $u(y_j^{s-1}) - u(y_j^s) \geq u(y_i^s) - u(y_i^{s-1})$ by strict increasingness and concavity of f. So, we have $y^{s-1} R_u y^s$ for each step $s = 1, 2, \ldots, m-1$. Hence, by transitivity, it follows that $y' R_u y$. □

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