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EXPOSITA NOTE

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Extreme inequality aversion without separability

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Abstract Hammond (J Econ Theory 11, 465–467, 1975), Meyer (J Econ Theory 11, 119–132, 1975), and Lambert (The distribution and redistribution of income Manchester University Press, Manchester, 2001) provide the formal result connecting leximin and the idea of extreme inequality aversion for social preferences of the expected utility type. Using an analogous approach, we show that for social preferences not necessarily satisfying the separability axiom that underlies expected utility theory, the case of extreme inequality aversion is covered by the class of *weakly maximin* social preferences—i.e., the class of social preferences that give priority to the worst off in all cases in which the worst off is not indifferent.

Keywords Inequality aversion · Leximin · Maximin · Risk aversion

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1 Introduction

For social preferences that can be represented by social welfare functions of the expected utility form, it is broadly accepted that leximin constitutes the case of

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extreme inequality aversion. As far as we know, the only formal justification for this connection between leximin and the idea of extreme inequality aversion is a result that can be attributed to Hammond (1975), Meyer (1975), and Lambert (2001). Loosely speaking, their result says that the choices over social alternatives implied by leximin coincide with the choices implied by all *most inequality averse* members (in the Arrow–Pratt sense) of the class of expected utility type social preferences.

Our contribution is to study the idea of extreme inequality aversion for social preferences that do not necessarily satisfy the separability axiom underlying expected utility theory. This more general outlook is very common in the social choice literature: see for instance the standard overviews of d'Aspremont and Gevers (2002) and Bossert and Weymark (2004).[1] We show that for these more general social preferences, the case of extreme inequality aversion is covered by the class of *weakly maximin* social preferences—these are all social preferences that have in common the property that they strictly prefer a given alternative over another if the worst off is strictly better off in the given alternative.[2] To establish this result, we prove an analogue of the Hammond–Meyer–Lambert result mentioned above.

2 Preliminaries

The set of individuals comprising society is $N = \{1, 2, \dots, n\}$ where n is a finite number. Let Y be a bounded subset of $\mathbb{R}_{++}$, and let $X = Y^n$. A social alternative is a vector $x = (x_1, x_2, \dots, x_n) \in X$ where x_i measures the well-being of individual $i \in N$. We use the symbol 1_n to denote an n-dimensional vector of which all components are equal to 1. For each $x \in X$, $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ denotes a rearrangement of x such that $\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_n$. For each $x, y \in X$, we write $x > y$ if and only if $x_i \geq y_i$ for each $i \in N$ with at least one inequality holding strictly. Social preferences are represented by a relation R ('is at least as good as') on X. It is assumed throughout that social preference relations are orderings.[3] We denote the asymmetric and symmetric parts of R by P ('is better than') and I ('is equally good as'), respectively.

The Hammond–Meyer–Lambert result applies to social preferences that are formally similar to the preferences of expected utility maximizers. We say that R is a member of this class, which we denote by $\mathcal{W}_{EU}$, if and only if it satisfies the following four well known properties:[4]

(i) *Anonymity*: For each $x \in X$, if x' is a rearrangement of x, then xIx'.
(ii) *Continuity*: For each $x \in X$, $\{y \in X \mid yRx\}$ and $\{y \in X \mid xRy\}$ are closed in X.

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[1] Likewise, in the literature on individual decision under risk, several non-separable alternatives to expected utility have been studied in order to accommodate for the empirical shortcomings of expected utility theory. For overviews, see Camerer (1995) and Starmer (2000).

[2] In their respective analyses concerning extreme inequality aversion, Bosmans (2005) and Tungodden and Vallentyne (2005) both arrive at the class of weakly maximin social preferences. However, both use approaches that deviate from the standard Arrow–Pratt approach.

[3] An ordering is a reflexive, transitive, and complete relation.

[4] See Bossert and Weymark (2004) for a discussion of these four properties. Their Theorem 13.5 shows that social preferences satisfying these properties are representable by a social welfare function of the expected utility form.

(iii) *Strong Pareto*: For each $x, y \in X$, if $x > y$, then xPy.

(iv) *Separability*: For each $\tilde{N} \subset N$ and each $x, y, x', y' \in X$,

$$[\forall i \in \tilde{N}, x_i = y_i, x'_i = y'_i, \text{and}, \forall i \in N \setminus \tilde{N}, x_i = x'_i, y_i = y'_i] \\ \Rightarrow [xRy \Leftrightarrow x'Ry'].$$

The result we shall present deals with social preferences that satisfy the basic properties (i) to (iii), but not necessarily separability. We use the symbol $\mathcal{W}$ to denote this more general class of social preferences.

The Arrow–Pratt concept was designed originally to compare members of $\mathcal{W}_{EU}$ with respect to inequality aversion. However, one of the several equivalent formulations of the concept—that based on the 'equally distributed equivalent well-being'—can be applied to the more general class $\mathcal{W}$ as well. For a social preference relation $R \in \mathcal{W}$ and for a social alternative $x \in X$, the equally distributed equivalent well-being, denoted by $\xi(R; x)$, is the per capita level of well-being that, when equally distributed, yields the same level of welfare as x according to R. Formally, for each $R \in \mathcal{W}$ and each $x \in X$, $\xi(R; x) = e$ if and only if $e1_n Ix$. The function $\xi(R; \cdot)$ on X is defined for each $R \in \mathcal{W}$, and is, moreover, a representation of R on X. The Arrow–Pratt concept says the following: for each $R, R' \in \mathcal{W}$, R is at least as inequality averse as R', which is written as $R \succeq_{AP} R'$, if and only if $\xi(R; x) \leq \xi(R'; x)$ for each $x \in X$.

The Hammond–Meyer–Lambert result shows that leximin—which gives priority to the worst off in a lexicographical fashion—can be interpreted as being extremely inequality averse with respect to the class $\mathcal{W}_{EU}$. Leximin, which we denote by R_{lex}, is defined as follows: for each $x, y \in X$,

$$xR_{lex}y \Leftrightarrow [x \text{ is a rearrangement of } y, \text{ or}, \exists k \in N, \forall i < k, \hat{x}_i = \hat{y}_i \text{ and } \hat{x}_k > \hat{y}_k].$$

Leximin is a member of the class of weakly maximin social preferences. The members of this class give priority to the worst off in all cases in which the worst off is not indifferent. Formally, a social preference relation R is weakly maximin if and only if, for each $x, y \in X$,

$$\hat{x}_1 > \hat{y}_1 \Rightarrow xPy.$$

It can be shown that leximin is the only social preference relation that is both weakly maximin and separable. We denote the class of anonymous and strongly Paretian weakly maximin social preferences by $\mathcal{M}$. Since $\mathcal{M}$ does not contain any continuous members, the classes $\mathcal{M}$ and $\mathcal{W}$ are disjoint.[5] Clearly, leximin is a member of $\mathcal{M}$. We wish to emphasize, however, that $\mathcal{M}$ contains as well social preferences that are very different from leximin. To see this, note that for each anonymous and strongly Paretian social preference relation R' there exists an R belonging to $\mathcal{M}$ such that, for each $x, y \in X$,

$$\hat{x}_1 = \hat{y}_1 \Rightarrow [xRy \Leftrightarrow xR'y].$$

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[5] Each member of $\mathcal{M}$ is a positional dictatorship and positional dictatorships cannot satisfy both continuity and strong Pareto (see Bossert and Weymark 2004, p. 1114). In relation to this, note that maximin—which implies social indifference for all pairs of social alternatives for which the worst off individuals are equally well off—is the only continuous member of the class of weakly maximin social preferences, but that it is not strongly Paretian and hence does not belong to $\mathcal{M}$.

Loosely speaking, this means that in all cases in which the worst off are equally well off, all social preferences are allowed as long as they satisfy anonymity and strong Pareto.

In the next section we provide a result, analogous to the Hammond–Meyer–Lambert result, which shows that if we broaden our attention from only the separable members of $\mathcal{W}$ (i.e., the members of $\mathcal{W}_{EU}$) to all members of $\mathcal{W}$, then the corresponding class of extremely inequality averse social preferences broadens from only the separable members of $\mathcal{M}$ (i.e., the single member of $\{R_{lex}\}$) to all members of $\mathcal{M}$.

3 Result

Relying heavily on Hammond (1975) and Meyer (1975), Lambert (2001, Theorem 4.4) presents a result that justifies the interpretation of leximin as extremely inequality averse, provided that separability is demanded.[6]

Theorem 0 *For each* $x, y \in X$ *such that* $\hat{x} \neq \hat{y}$*,*

$$xP_{lex}y \;\Leftrightarrow\; [\exists R \in \mathcal{W}_{EU},\; \forall R' \in \mathcal{W}_{EU},\;\; R' \succeq_{AP} R \;\Rightarrow\; xR'y].$$

Theorem 0 says that (i) whenever leximin implies a strict preference over a pair of social alternatives, then the most inequality averse social preferences in the class $\mathcal{W}_{EU}$ unanimously agree with that preference (with weak preference), and, conversely, (ii) whenever all most inequality averse members of $\mathcal{W}_{EU}$ weakly prefer one social alternative over another, then leximin agrees with this preference (with strict preference).

Our contribution is to provide the following corresponding result for the case where separability is not demanded.

Theorem 1 *For each* $x, y \in X$ *such that* $\hat{x} \neq \hat{y}$*,*

$$[\forall R \in \mathcal{M},\;\; xPy] \;\Leftrightarrow\; [\exists R \in \mathcal{W},\; \forall R' \in \mathcal{W},\;\; R' \succeq_{AP} R \;\Rightarrow\; xR'y].^{7}$$

Proof ($\Rightarrow$) Take arbitrary $x, y \in X$ such that $\hat{x} \neq \hat{y}$ and

$$\forall R \in \mathcal{M}, \qquad xPy.$$

The latter ensures that $\hat{x}_1 > \hat{y}_1$. We have to show that there exists an $R \in \mathcal{W}$ such that

$$\forall R' \in \mathcal{W}, \qquad R' \succeq_{AP} R \;\Rightarrow\; xR'y. \tag{1}$$

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[6] Hammond (1975) proves the result only for a proper subclass of $\mathcal{W}_{EU}$, viz., for social preferences of the constant-elasticity-of-substitution type. Lambert (2001) uses the analysis of Meyer (1975) to extend Hammond's result to the entire class $\mathcal{W}_{EU}$.

[7] An alternative to Theorem 1 which is closer to Hammond's (1975) original formulation can as well be obtained. This requires a somewhat stronger concept of inequality aversion: R is strongly more inequality averse than R', which is written as $R \succ_{SAP} R'$, if and only if $\xi(R; x) < \xi(R'; x)$ for each $x \in X$ with at least two distinct components (note that this version of the Arrow–Pratt concept is more demanding than the asymmetric part of $\succeq_{AP}$). The alternative to Theorem 1 is: for each $x, y \in X$, $[\forall R \in \mathcal{M},\; xPy] \Leftrightarrow [\exists R \in \mathcal{W},\; \forall R' \in \mathcal{W},\; R' \succ_{SAP} R \Rightarrow xP'y]$. We omit the proof because it is very similar to that of Theorem 1.

Consider an $R'' \in \mathcal{W}$ such that $\xi(R''; x) \leq \xi(R''; y)$. If such an R'' does not exist, then each member of $\mathcal{W}$ can serve as an R for which (1) holds. If, on the other hand, such an R'' exists, then we define R such that, for each $w, z \in X$,

$$wRz \Leftrightarrow \alpha \hat{w}_1 + (1-\alpha)\xi(R''; w) \geq \alpha \hat{z}_1 + (1-\alpha)\xi(R''; z),$$

where $\alpha \in [0, 1)$ is such that

$$\alpha \hat{y}_1 + (1-\alpha)\xi(R''; y) = \hat{x}_1.$$

By strong Pareto and reflexivity, $\xi(R''; x) \geq \hat{x}_1$, and, hence, we have $\xi(R''; y) \geq \hat{x}_1$. The latter, combined with the fact that $\hat{x}_1 > \hat{y}_1$, ensures that R can be defined in the above way. It can be readily checked that $R \in \mathcal{W}$. Since $\xi(R''; \hat{x}_1 1_n) = \hat{x}_1$, we have $\hat{x}_1 1_n I y$ and, consequently, $\xi(R; y) = \hat{x}_1$.

What remains to be shown is that (1) is true for the constructed R. For each R' such that $R' \succeq_{AP} R$, we have

$$\hat{x}_1 = \xi(R; y) \geq \xi(R'; y),$$

and, by strong Pareto and reflexivity, $\xi(R'; x) \geq \hat{x}_1$. By consequence, we have $\xi(R'; x) \geq \xi(R'; y)$, and so $xR'y$.

($\Leftarrow$) Seeking a contradiction, we assume there exist $x, y \in X$ such that $\hat{x} \neq \hat{y}$ and

$$\exists R \in \mathcal{W}, \quad \forall R' \in \mathcal{W}, \qquad R' \succeq_{AP} R \ \Rightarrow \ xR'y, \tag{2}$$

while there exists an $R'' \in \mathcal{M}$ such that $yR''x$. Note that from $yR''x$ it follows that $x > y$ does not hold by strong Pareto, and also that $\hat{x}_1 \leq \hat{y}_1$.

Define $R''' \in \mathcal{W}$ such that, for each $w, z \in X$,

$$wR'''z \ \Leftrightarrow \ \frac{\sum_{i=1}^n \phi_i \hat{w}_i}{\sum_{i=1}^n \phi_i} \geq \frac{\sum_{i=1}^n \phi_i \hat{z}_i}{\sum_{i=1}^n \phi_i},$$

where $\phi_1, \phi_2, \ldots, \phi_n > 0$. The weights are determined in two steps. First, we choose arbitrary $\phi_1, \phi_2, \ldots, \phi_n > 0$ such that $yP'''x$. A choice of the weights such that $yP'''x$ is possible since $x > y$ does not hold. Second, we increase the weight ϕ_1 while holding all other weights fixed until $\xi(R'''; w) \leq \xi(R; w)$ for each $w \in X$. This is possible because, by choosing ϕ_1 sufficiently high,

$$\xi(R'''; w) = \frac{\sum_{i=1}^n \phi_i \hat{w}_i}{\sum_{i=1}^n \phi_i}$$

can be chosen as close to $\hat{w}_1$ as necessary for each $w \in X$ since X is bounded. Note that increasing ϕ_1 while holding all the other weights fixed preserves the ranking $yP'''x$ since $\hat{x}_1 \leq \hat{y}_1$.

Now, because $R''' \in \mathcal{W}$ is such that $\xi(R'''; w) \leq \xi(R; w)$ for each $w \in X$, and, moreover, $yP'''x$, we obtain a contradiction of (2). □

Theorem 1 is completely analogous to Theorem 0: (i) if all members of the class $\mathcal{M}$ imply a strict preference over a pair of social alternatives, then the most inequality averse social preferences in the general class $\mathcal{W}$ unanimously agree with that preference (with weak preference), and, conversely, (ii) if all most inequality averse members of $\mathcal{W}$ weakly prefer one social alternative over another, then the members of $\mathcal{M}$ unanimously agree with this preference (with strict preference).

References

d'Aspremont, C., Gevers, L.: Social welfare functionals and interpersonal comparability. In: Arrow, K.J., Sen, A.K., Suzumura, K. (eds.) Handbook of social choice and welfare, vol. 1. Amsterdam: Elsevier (2002)

Bosmans, K.: Comparing degrees of inequality aversion. CES Discussion Paper 05.17, Katholieke Universiteit Leuven (2005)

Bossert, W., Weymark, J.A.: Utility in social choice. In: Barberà, S., Hammond, P.J., Seidl, C. (eds.) Handbook of utility theory, vol. 2, Extensions. Dordrecht: Kluwer (2004)

Camerer, C.: Individual decision making. In: Kagel, J.H., Roth, A.E. (eds.) The handbook of experimental economics. Princeton: Princeton University Press (1995)

Hammond, P.J.: A note on extreme inequality aversion. J. Econ. Theory **11**, 465–467 (1975)

Lambert, P.J.: The distribution and redistribution of income, 3rd edn. Manchester: Manchester University Press (2001)

Meyer, J.: Increasing risk. J. Econ. Theory **11**, 119–132 (1975)

Starmer, C.: Developments in non-expected utility theory: the hunt for a descriptive theory of choice under risk. J. Econ. Lit. **38**, 332–382 (2000)

Tungodden, B., Vallentyne, P.: On the possibility of Paretian egalitarianism. J. Phil. **102**, 126–154 (2005)