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New perspectives on a more-or-less familiar poverty index

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Abstract In this article, we examine some properties and extensions of a particular poverty index, which is already passingly familiar to poverty analysts. Our investigation leads to a new perspective on the family of additively decomposable and scale-invariant poverty indices and provides new insights into the class of poverty measures proposed by Foster et al. (Econometrica 52:761–766, 1984). Finally, our work leads to a clarification of the different roles of transfer properties and distribution sensitivity in poverty measurement.

1 Introduction

Two issues on which the research on poverty measurement has focused are the behaviour of poverty indices when income values and the poverty line are multiplied by a positive scalar, and when income transfers take place among the poor. The *scale invariance axiom* asks for a measure of poverty to remain unchanged after scaling all incomes and the poverty line by a common factor. Different specifications of the *transfer axiom* are based on the general idea that non-reranking transfers among the

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poor should increase the poverty index if the donors are poorer than the recipients, at least if the latter do not cross the poverty line as a consequence of the transfer. There is apparently no relationship between the property of scale invariance and versions of the transfer principle.

We show that the individual poverty contributions of all additively decomposable and scale-invariant poverty indices are, in fact, transformations of a specific one resulting from the sum ad infinitum of all moments of the distribution of the normalised poverty gaps. By virtue of this feature, the building block of these indices is an individual poverty function enjoying the highest order transfer property, upper unboundedness and discontinuity at any finite poverty line. Further, the correspondence between moments of the distribution of the normalised poverty gaps and members of the class of poverty indices of Foster et al. (1984) enhances our result, yielding a deeper understanding of this widely used class. Finally, by making recourse to the parametric formulation of converging series, we derive extended families of measures, which are helpful in illustrating the concepts of *transfer properties* and *distribution sensitivity* in poverty measurement, as proposed by Zheng (2000) and Chiu (2007).

The article develops as follows. In Sect. 2, we lay out our notation and present the poverty index which is the starting point of our analysis. Section 3 illustrates the notions of transfer properties and distribution sensitivity and presents the behaviour of this index with respect to them. In Sect. 4, we offer our main result. In Sect. 5, we use parametric extensions of our main result to illustrate transfer properties and distribution sensitivity in poverty measurement.

2 The more-or-less familiar poverty index

In a society of n individuals, let $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_{++}$ be the vector of incomes arranged in non-decreasing order, where y_i is the income of the ith individual, and let $z \in \mathbb{R}_{++}$ be an exogenously given poverty line. The individuals i for whom $y_i < z$ are the poor ones. Let y_q be the largest poor income, so that the headcount ratio is $H = q/n$. The poverty index we are interested in, which we denote P^∞, takes the form:

$$P^\infty(y; z) = \frac{1}{n} \sum_{i=1}^{q} \frac{z}{y_i}.$$

The reason for this notation will become apparent later.

The index P^∞ may not appear to be very familiar, but it has antecedents which involve what Chakravarty (1983, p. 81) describes as 'a fairly natural translation of a relative inequality index of a censored income distribution into a relative poverty index'.[1] Moreover, the poverty contribution function inherent in P^∞ forms a building

[1] The reader may verify that P^∞/zH is the reciprocal of the equally distributed equivalent poor income for the Atkinson (1970) utility of income function $U(x) = -1/x$. Moreover, a member of the second family of poverty indices suggested by Clark et al. (1981) is a simple function of P^∞ and the headcount ratio. This arises when welfare over basic incomes $b_i = \min\{y_i, z\},\ 1 \le i \le n$, is defined in terms of the utility function $U(x) = -1/x$.

block for the entire class of additively decomposable and scale-invariant indices of poverty, $P(y; z)$, which evaluate aggregate poverty as a normalised sum of individual poverty contributions $p(y_i; z)$ defined by:

$$P(y; z) = \frac{1}{n} \sum_{i=1}^{n} p(y_i; z), \tag{1}$$

where $p(y_i; z) = 0$ if $y_i \geq z$ and $p(y_i; z) > 0$ otherwise, $p(y_i; z) = p(\lambda y_i; \lambda z)$ for all $\lambda > 0$ (scale invariance), and $p(y_i; z)$ is continuous and non-increasing in y_i for $y_i \in (0, z)$. The index P^{∞} is in this form, as is the Watts (1968) index, for which $p(y_i, z) = \ln(z/y_i)$, and the so-called 'FGT index' of Foster et al. (1984), call it P_α, for which $p(y_i; z) = g_i^{\alpha}$, where $g_i = (z - y_i)/z$ is person i's normalized poverty gap and $\alpha \in \mathbb{N} \cup 0$.[2]

The individual poverty contributions of all such indices are, in fact, transformations of the poverty contribution of P^{∞}. For as Foster and Shorrocks (1991) show, for a poverty index satisfying (1) and the accompanying restrictions,

$$p(y_i; z) = \varphi\left(\frac{z}{y_i}\right) \tag{2}$$

for some continuous and non-decreasing function φ. The 'building block' for the class is thus the individual poverty function z/y_i of P^{∞}. In Fig. 1, $\varphi(z/y_i)$ is plotted as a continuous line when φ is the identity function, namely, for P^{∞}, and dotted lines show the pattern of values $\varphi(z/y_i)$ takes for the Watts index and for P_α, $\alpha = 0, 1, 2, 3$. The function φ in (2) can be seen as a transformation apt to choose which properties inherent in the hyperbolic functional form z/y_i are desired, and which not, in an index in the class defined by (2). The specification $\varphi(z/y_i) = z/y_i$ enjoys upper unboundedness and discontinuity at any finite poverty line. The property of upper unboundedness is rejected for all members of the P_α class but is accommodated by the Watts index; poverty line discontinuity is retained by P_α for $\alpha = 0$ but not by the other members of the P_α class, nor by the Watts index.[3]

3 Transfer properties, distributional sensitivity and P^{∞}

In order to illustrate the characteristics of P^{∞} in terms of *transfer properties* and *distribution sensitivity*, let us consider individual poverty contribution functions $p(y_i; z)$ in (1) that are (infinitely) differentiable and monotonically decreasing in y_i for $y_i \in (0, z)$.

Transfer properties concern the directional change signalled by a poverty index as a consequence of the redistribution of poor incomes. In what follows we assume that no income crosses the poverty line as a result of such redistribution. The property

[2] Although Foster et al. (1984) do not specify the nature of the parameter α, thus allowing it to be taken as any non-negative real number, values for the parameter are usually drawn from the set of non-negative integers, as we shall assume in this article.

[3] Note that if zero incomes are admitted, then both P^{∞} and the Watts index are undefined.

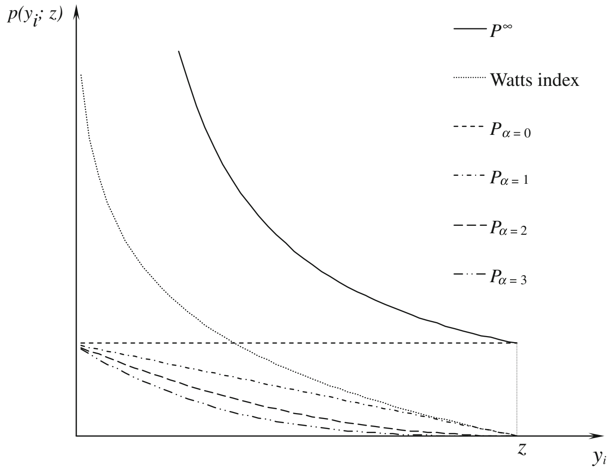

Fig. 1 Poverty contribution functions

introduced in the poverty literature by Sen (1976) with the label *transfer axiom* is based on the idea that a disequalizing transfer increases poverty. Its accommodation in our context requires that $\partial^2 p/\partial y_i^2 > 0$. The interest in higher degrees of transfer properties began with Kolm (1976a,b) in the field of inequality measurement and was introduced into the poverty literature by Kakwani (1980) with the formulation of the *transfer sensitivity axiom*. The idea is that a poverty increase should be the net effect of an equalizing and a disequalizing transfer of equal magnitudes if the latter takes place further down in the income distribution. The restriction on $p(y_i; z)$ is in this case that $\partial^3 p/\partial y_i^3 < 0$.

Elegant theory for transfer properties of higher orders is fully expounded by Fishburn and Willig (1984), and essentially postulates that 'any combination of a socially desirable transfer [or series of transfers] with its inverse at uniformly higher levels of income will have positive social benefit' (p. 323). It is well known that successively higher orders of transfer properties are linked with welfare functions whose derivatives alternate in sign. In our framework, the requirements for the accommodation of higher orders of transfer properties follow the scheme $\partial^\beta p/\partial y_i^\beta < 0$ for $\beta = 3, 5, 7, \ldots$ and $\partial^\beta p/\partial y_i^\beta > 0$ for $\beta = 2, 4, 6, \ldots$. It is easy to verify that the poverty index P^∞ satisfies transfer properties of *all* orders—up to the *infinite* order, hence our notation. Here is a link, then, between scale invariance and transfer properties for additively decomposable poverty indices, unremarked upon in previous literature. The transformation function φ in (2) determines which transfer properties are satisfied by an index in the class defined by (1), just as it conditions the upper (un)boundedness and poverty line (dis)continuity properties of the index. More specifically, the role of φ can now be interpreted as potentially lowering the order at which transfer properties are met by the index in (1) vis-à-vis the infinite order of P^∞.

Whilst transfer properties require that a poverty index react to a transfer or combination of transfers, the concept of *distribution sensitivity* quantifies *how responsive* an index is to such transfers. The Zheng (2000) measure of distribution sensitivity deals with an equalizing transfer among two poor individuals, as in the transfer axiom. The Chiu (2007) measure deals with the combination of an equalizing and a disequalizing transfer, as in the transfer sensitivity axiom. Both measures take the form

$$s_p^k(y_i; z) = -\frac{\partial^k p/\partial y_i^k}{\partial^{k-1} p/\partial y_i^{k-1}} \quad \text{for } y_i < z, \tag{3}$$

where k is the order of the relevant transfer property. The Zheng measure is obtained if $k = 2$ and the Chiu measure if $k = 3$. Higher orders of distribution sensitivity measures can also be envisaged.

For the P^∞ index, we find $s_{p_\infty}^k(y_i; z) = k/y_i$. Hence, we have $s_{p_\infty}^2(y_i; z) = 2/y_i$ for the Zheng measure and $s_{p_\infty}^3(y_i; z) = 3/y_i$ for the Chiu measure. Again, the transformation φ determines how an index deviates from the building block index P^∞. For example, the Zheng measure for the general poverty index in (1), with poverty contribution function $p(y_i; z) = \varphi(z/y_i)$ as in (2), gives

$$s_p^2(y_i; z) = s_{p_\infty}^2(y_i; z) + \frac{z}{y_i^2}\frac{\varphi''(z/y_i)}{\varphi'(z/y_i)} \quad \text{for } y_i < z,$$

showing how its distribution sensitivity relative to that of the building block index P^∞ is conditioned by the concavity/convexity property of φ.

4 A result which links P^∞ with the class of FGT indices

A simple application of the theory of the convergent geometric series links P^∞ firmly with the P_α class of Foster et al. (1984). Recalling that $g_i = (z - y_i)/z$ is person i's normalized poverty gap, $i < q$, and that g_i^α is that person's contribution to the index P_α, observe that:

$$\frac{z}{y_i} = \left(1 - \frac{z - y_i}{z}\right)^{-1} = \frac{1}{1 - g_i} = 1 + g_i^1 + g_i^2 + g_i^3 + \cdots = \lim_{M\to\infty}\sum_{\alpha=0}^{M} g_i^\alpha.$$

The poverty contribution function inherent in P^∞ is thus the *infinite sum* of those inherent in the FGT indices P_α across integer values of α. In turn, one can write P^∞ itself as an infinite sum of these FGT indices:

$$P^\infty = \frac{1}{n}\sum_{i=1}^{q}\frac{z}{y_i} = \lim_{M\to\infty}\sum_{\alpha=0}^{M}\frac{1}{n}\sum_{i=1}^{q} g_i^\alpha = \lim_{M\to\infty}\sum_{\alpha=0}^{M} P_\alpha.$$

We can thus think of the FGT class as actually providing the *building block class* for all scale-invariant and additively decomposable poverty indices as in (2).

It is also instructive to compare the distribution sensitivity of P^∞ with that of P_α in terms of the Zheng and Chiu measures. For P_α, we find $s^k = (\alpha - k + 1)/(z - y_i)$. Hence, for the Zheng measure, we have $s^2 = (\alpha - 1)/(z - y_i)$ and for the Chiu measure we have $s^3 = (\alpha - 2)/(z - y_i)$. Thus, whilst P^∞ has declining distribution sensitivity, P_α has increasing distribution sensitivity.[4] The index P^∞ is more distribution-sensitive at low (poor) income levels, whilst the index P_α is more so at high (poor) income levels. In terms of s^k, the crossover point for P^∞ and P_α is at $y_\alpha = kz/(\alpha + 1)$ which, of course, approaches zero as $\alpha \to \infty$.

5 Parametric extensions of P^∞

Here, we introduce two extended families of poverty indices belonging to the class (1). We shall call them P_γ^∞ and $P_{\gamma,\omega}^\infty$ where γ and $\omega > \gamma$ are non-negative integers. Just as P^∞ has been revealed to be the sum ad infinitum of the FGT indices P_α for integer values of α, these new indices are *truncated* sums of FGT indices, the one an infinite sum and the other a finite sum.

The aggregate poverty index P_γ^∞ relies on the individual deprivation function p_γ^∞, which results from a parameterized version of the algebra of converging series that led to p^∞:

$$p_\gamma^\infty(y_i; z) = \frac{g_i^\gamma}{1 - g_i} = g_i^\gamma + g_i^{\gamma+1} + g_i^{\gamma+2} + g_i^{\gamma+3} + \cdots = \lim_{M \to \infty} \sum_{\alpha=\gamma}^{M} g_i^\alpha. \quad (4)$$

The reader can note that $p_\gamma^\infty(y_i; z) = g_i^\gamma \times (z/y_i)$ is the product of an FGT poverty contribution function, that inherent in P_γ, and the poverty contribution function of P^∞. The aggregate poverty measure corresponding to (4) is an infinite sum of FGT indices, beginning with P_γ:

$$P_\gamma^\infty(y; z) = \lim_{M \to \infty} \sum_{\alpha=\gamma}^{M} P_\alpha(y; z) = P_\gamma(y; z) + P_{\gamma+1}(y; z) + \cdots .$$

The integer γ is the order of transfer property enjoyed by the summand that satisfies transfer properties up to a lower order than any other summand.

The further extended class of poverty indices $P_{\gamma,\omega}^\infty(y; z)$ emerges as a finite sum of FGT indices:

$$\begin{aligned} P_{\gamma,\omega}^\infty(y; z) &= \frac{1}{n} \sum_{i=1}^{n} p_{\gamma,\omega}^\infty(y_i; z) = \frac{1}{n} \sum_{i=1}^{n} \left[p_\gamma^\infty(y_i; z) - p_\omega^\infty(y_i; z) \right] \\ &= P_\gamma(y; z) + P_{\gamma+1}(y; z) + \cdots + P_{\omega-1}(y; z). \end{aligned} \quad (5)$$

It can be noted that (i) for $\omega = \gamma + 1$ the members of the P_α class are generated, and (ii) for $\omega \to \infty$ we get the members of the P_γ^∞ class.

[4] As Zheng (2000, p. 123) points out, satisfaction of the transfer sensitivity axiom is a necessary but not a sufficient condition for a poverty index to exhibit diminishing distribution sensitivity.

The new classes of indices permit to draw a clearer distinction between transfer properties and distribution sensitivity. The index $P^{\infty}_{\gamma,\omega}$ takes the highest order transfer property of the *final* FGT measure in sum (5), while its distribution sensitivity is an average of the distribution sensitivities of *all* the FGTs in the sum. Irrespective of the value of γ, the index $P^{\infty}_{\gamma,\omega}$ satisfies transfer properties of all orders up to $\omega - 1$. Both γ and ω positively affect the values of the Zheng and Chiu measures of distribution sensitivity.[5] Hence, for a given value of ω, changing the value of γ changes the degree of distribution sensitivity without affecting the highest order transfer property satisfied. The measure $P^{\infty}_{\gamma,\omega}$ thus allows to some extent to combine relatively high degrees of transfer principle with relatively low degrees of distribution sensitivity, or vice versa. The simple FGT indices do not allow this much flexibility, nor do any other poverty indices we know of.

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[5] Whether or not distribution sensitivity increases *strictly* if an FGT index is removed (by increasing γ) or added (by increasing ω) in sum (5) depends on whether or not the derivatives in (3) are zero for the FGT index in question. For example, while the value of the Zheng measure is higher for P^{∞}_{2} than for P^{∞}, the value of the Chiu measure is the same for the two indices.