Cost sharing in the capacity synthesis problem

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February 2007

Extended abstract:
Cooperative game theory brings fairness and incentive compatibility concepts to the analysis of combinatorial cost sharing problems. The earliest and most studied example is the minimal cost spanning tree problem (thereafter mcst). The related *capacity synthesis* problem has more potential for economic applications, yet has not been discussed by the game theory literature. The n agents share a network for bilateral exchange of information, transportation of commodities along roads or shipping channels, distributing utilities along a grid, etc... The traffic between agents i and j requires a certain capacity t_{ij} (bandwidth, width of a road, depth of a channel, ..). The problem is to choose a minimal cost graph such that any pair i, j is connected by a path of which each edge has capacity at least t_{ij}. We assume that the cost of capacity is identical accross edges.

The mcst and the capacity synthesis problems are closely related: in the latter an optimal network is the spanning tree with *maximal* cost with respect to the capacity matrix $t = [t_{ij}]$.

In both problems core stability is the central incentive and fairness property. In the capacity problem there are *two* interpretations of the stand alone cost of a coalition S. The *upper stand alone* test is the familiar incentive property. It says that S should not be charged *more* than $v_+(S, t)$, the minimal cost of serving all traffic demands of S. The *lower stand alone* test is a fairness property. It says S should not be charged *less* than $v_-(S, t)$, the minimal cost of serving the traffic between members of S only. Two other minimal requirements are i) *Continuity* : cost shares y_i depend continuously upon t, and ii) *Monotonicity* (MON): y_i is weakly increasing in t_{ij} for all j.

We discuss two interpretations of the model. If users are not responsible for their capacity requests, the *Solidarity* axiom require y_i to be weakly increasing in all requests t_{jk}, so that i does not take advantage of other users' increased "needs". This very powerful requirement is introduced by Bergantinos and Vidal-Puga in the mcst problem. Here too it implies *Population Monotonicity*: cost shares y_i weakly increase when we add a new agent. Moreover a solidary solution is *reductionist.*

The *irreducible* capacity matrix t^* is the largest matrix weakly larger than t and with the same optimal cost as t. A solution is reductionist if it only depends upon the irreducible matrix t^*. An advantage of reductionist solutions is that they are immune to misreport of traffic needs: if the optimal network sustains a larger capacity between i and j than t_{ij}, under-reporting t_{ij} would not change the optimal newtork but may reduce the cost share of $\{i, j\}$. This cannot happen with a reductionist solution.

The two cooperative games $(N, v_{-}(\cdot, t^*))$ and $(N, v_{+}(\cdot, t^*))$ have the same Shapley value, that we call the Shapley* solution. This solidary solution is the counterpart of that proposed by Bergantinos and Vidal-Puga in the mcst problem. It has a computationally simple closed form expression.

The key critique of reductionist solutions is that they erase important differences in individual capacity requests. Consider the situation where agent 1 is a communication "hub": $t_{1i} = 1$ for all $i \geq 2$, and $t_{ij} = 0$ for all $i, j \geq 2$. As the matrix t^* is fully symmetric ($t^*_{ij} = 1$ for all i, j), under any reductionist solution agent 1 pays the same as everyone else. If users are minimally responsible for their requests, agent 1 *should* be charged more than other agents (typically half of total costs under the solutions discussed below).

In the alternative interpretation of our model where users are held responsible for their capacity requests, a minimal requirement is the *Ranking* property: if $t_{ik} > t_{jk}$ for all $k \neq i, j$, then $y_i > y_j$. The counterpart of Solidarity is *Cross Monotonicity:* y_i must be weakly *de*creasing in all requests t_{jk}. Finally the *Equal Responsibility* property says that if t_{12} is the only communication need between two complementary sets of agents, this cost should be shared equally between users 1 and 2.

The Shapley value of the cooperative game $(N, v_{-}(\cdot, t))$, as well as that of the game $(N, v_{+}(\cdot, t))$, meet these three properties, but fail the lower stand alone test. Moreover they are not computationally tractable (exponential). Another solution meeting Ranking, Cross monotonicity and Equal Responsibility is the $Bird^{\frac{1}{2}}$ solution. It charges to agent i one half of the cost t_{ij} of each edge ij adjacent to i on the maximal cost spanning tree (averaging in case there is more than one such tree). It keeps the cost share of S between $v_-(S,t)$ and $\frac{1}{2}(v_-(S,t) + v_+(S,t))$, thus tightening significantly the upper stand alone bound. Its computation is polynomial when there is a single maximal spanning tree, not so when there is more than one. Moreover in that case cost shares are not continuous in t.

We define a continuous solution $BM^{\frac{1}{2}}$, that coincides with $Bird^{\frac{1}{2}}$ when $t_{ij} = 0$ or 1 for all i, j, and such that cost shares are piecewise linear w.r.t. t. It preserves all the above properties of $Bird^{\frac{1}{2}}$, and its computation remains hard.

A recent variation on core stability is *merge-proofness*: no coalition could benefit by building a network covering its *internal* communication needs, and posing as a single agent with appropriate capacity demands with all outsiders. Merging maneuvers are less dramatic than the full blown secession invoked in the upper stand alone test, hence they are easier to implement. Thus merge-proofness has typically more bite than core stability. An open question is the robustness of our two solutions, Shapley* and $BM^{\frac{1}{2}}$ to merging, as well to the reverse maneuver where a user *splits* into several aliases among whom he assigns his capacity needs.