Utility Proportional Beliefs in Games

Christian Bach & Andrés Perea

Maastricht University

EpiCenter

Rotterdam, February 2012

Motivation

- In epistemic game theory, the central idea is common belief in rationality or rationalizability.

  (Bernheim (1984), Pearce (1984), Brandenburger and Dekel (1987), Tan and Werlang (1988)).

- Key condition: A player always assigns probability zero to opponents' choices that are not optimal .

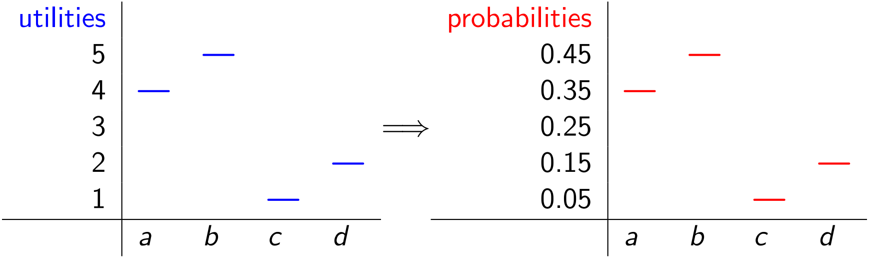
- In some games of interest, common belief in rationality does not match the experimental findings. See, for instance, the Traveler's Dilemma, and Guessing 2/3 of the Average .
- Possible explanation: In these games, people also assign positive probability to opponents' choices that are suboptimal .

In this paper, we explore the idea that

- a player assigns positive probability to every opponent's choice, but
- the probability assigned to an opponent's choice must be increasing in the expected utility generated by it.

More precisely: for every two opponent's choices *a* and *b*,

- the difference between the probabilities assigned to *a* and *b* must be proportional to the difference between the expected utilities generated by *a* and *b*.
- Utility proportional beliefs .



Relation to other concepts

The idea of utility proportional beliefs already appears in Rosenthal's t-solution (Rosenthal (1989)).

Two major differences:

- t-solution assumes that all players use same proportionality factor, we do not.
- t-solution assumes that everybody is correct about the players' actual beliefs, we do not.

Idea of utility proportional beliefs is also related to

- concept of quantal response equilibrium (McKelvey and Palfrey (1995)), and
- concepts of proper rationalizability (Schuhmacher (1999), Asheim (2001)) and proper equilibrium (Myerson (1978)) .

Epistemic Model

Consider a finite static game Γ where

- $I = \{1, ..., n\}$ is the finite set of players ,
- C_i is the finite set of choices for player i, and
- $u_i : C_1 \times ... \times C_n \to \mathbb{R}$ is the utility function for player i.

Definition (Epistemic model)

An epistemic model M for Γ specifies

(1) for all players i a set of types T_i, and

(2) for every type $t_i \in T_i$ some probabilistic belief $b_i(t_i)$ on $C_{-i} \times T_{-i}$ with finite support.

- From this epistemic model M, we can derive for every type t_i the complete belief hierarchy .

Utility proportional beliefs

- Fix some numbers $\lambda_{ij} \geq 0$ for every two distinct players i and j. Let $\lambda_i = (\lambda_{ij})_{j \neq i}$, and $\lambda = (\lambda_i)_{i \in I}$.

Definition (Utility proportional beliefs)

A type t_i has λ_i-utility proportional beliefs if for every opponent's type t_j it deems possible, and for every two opponent's choices c_j and c_j',

$$b_i(t_i)(c_j \mid t_j) - b_i(t_i)(c_j' \mid t_j) = \lambda_{ij} \cdot \left(u_j(c_j, t_j) - u_j(c_j', t_j)\right).$$

- The larger the proportionality factor λ_{ij}, the more dispersed the probabilities on j's choices will be.

Definition (Common belief in utility proportional beliefs)

(Induction start) A type t_i expresses 1-fold belief in λ-utility proportional beliefs if t_i only assigns positive probability to opponents' types t_j that hold λ_j-utility proportional beliefs.

(Inductive step) Let $k \geq 2$. Type t_i expresses k-fold belief in λ-utility proportional beliefs if t_i only assigns positive probability to opponents' types t_j that express $(k-1)$-fold belief in λ-utility proportional beliefs .

Type t_i expresses common belief in λ-utility proportional beliefs if t_i expresses k-fold belief in λ-utility proportional beliefs, for every k.

Algorithm

- We want to find those choices that are optimal for a type t_i that has λ_i-utility proportional beliefs, and expresses common belief in λ-utility proportional beliefs .
- Is there an algorithm that helps us find these choices?
- The algorithm will iteratedly remove beliefs – not choices!

Key insight:

- Suppose that player i has λ_i-utility proportional beliefs .
- Suppose player i believes that opponent j holds belief $b_j \in \Delta(C_{-j})$.
- Then, player i's belief about j's choices must be given by

$$p_{ij}(c_j) = \underbrace{\frac{1}{|C_j|} + \lambda_{ij} \cdot \left(u_j(c_j, b_j) - u_j^{average}(b_j) \right)}_{p^*_{ij}(b_j)\ :\ \text{utility proportional belief on } C_j \text{ induced by } b_j} .$$

Here,

$$u_j^{average}(b_j) = \frac{1}{|C_j|} \sum_{c_j \in C_j} u_j(c_j, b_j) \ : \ \text{average utility induced by } b_j$$

- Remember: For every belief $b_j \in \Delta(C_{-j})$, the induced λ_i-utility proportional belief $p^*_{ij}(b_j)$ is given by

$$p^*_{ij}(b_j)(c_j) = \frac{1}{|C_j|} + \lambda_{ij} \cdot \left(u_j(c_j, b_j) - u_j^{average}(b_j) \right)$$

for every $c_j \in C_j$.

Algorithm (Iterated elimination of disproportional beliefs)

(Induction start) For all players i, define the set of beliefs $B_i^0 := \Delta(C_{-i})$.

(Inductive step) For every $k \geq 1$, define the set of beliefs

$$B_i^k := \{ b_i \in \Delta(C_{-i}) : \ marg_{C_j} b_i \in p^*_{ij}(B_j^{k-1}) \ \text{ for all opponents } j \}.$$

- This algorithm is very easy to implement on a computer — even I could do it!

- Typically does not converge within finitely many steps!

Theorem (Algorithmic characterization)

The only beliefs that are possible for a type that has λ_i-utility proportional beliefs, and expresses common belief in λ-utility proportional beliefs, are the beliefs that survive iterated elimination of disproportional beliefs.

Algorithm (Iterated elimination of disproportional beliefs)

(Induction start) *For all players* i, *define the set of beliefs* $B_i^0 := \Delta(C_{-i})$.

(Inductive step) *For every* $k \geq 1$, *define the set of beliefs*

$$B_i^k := \{b_i \in \Delta(C_{-i}) : \; marg_{C_j} b_i \in p_{ij}^*(B_j^{k-1}) \; \text{ for all opponents } j\}.$$

Let λ_{ij}^{max} be maximal λ_{ij} such that $p_{ij}^*(b_j)$ is a well-defined belief for every $b_j \in \Delta(C_{-j})$.

Lemma (Unique beliefs for two players)

Consider a two-player game. If $\lambda_{12} < \lambda_{12}^{max}$ *and* $\lambda_{21} < \lambda_{21}^{max}$, *then for both players the set* B_i^k *converges to a unique belief vector.*

- Is no longer true for games with three players or more!

Theorem (Unique beliefs for two players)

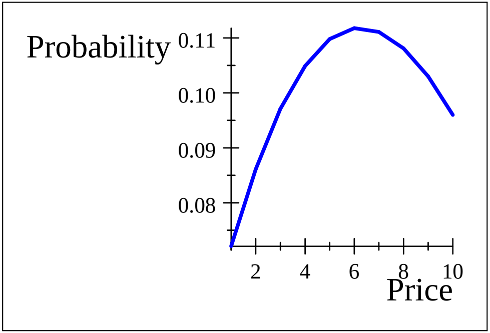
Consider a two-player game. If $\lambda_{12} < \lambda_{12}^{\max}$ and $\lambda_{21} < \lambda_{21}^{\max}$, then for both players there is a unique belief vector that he can hold under common belief in λ-utility proportional beliefs.

- This unique belief vector is very easy to compute with a computer, even for large games!

Traveler's Dilemma

Scenario: 10 prices, bonus = 2, penalty = 2.

- Common belief in rationality: Both players choose price 1.
- Common belief in utility proportional beliefs: Take $\lambda = \lambda^{\max}$.
  Unique belief of player i about j's choice is



Unique optimal choice is to choose price 6.

Matches well the experimental findings!

Thank you for your attention!

Any questions?

- G.B. Asheim, ‘Proper rationalizability in lexicographic beliefs’, *International Journal of Game Theory*, 30 (2001), 453–478
- B.D. Bernheim, ‘Rationalizable strategic behavior’, *Econometrica*, 52 (1984), 1007–1028
- A. Brandenburger and E. Dekel, ‘Rationalizability and correlated equilibria’, *Econometrica*, 55 (1987), 1391–1402
- R.D. McKelvey and T.R. Palfrey, ‘Quantal response equilibria for normal form games’, *Games and Economic Behavior* 10 (1995), 6–38
- R.B. Myerson, ‘Refinements of the Nash equilibrium concept’, *International Journal of Game Theory*, 7 (1978), 73–80

- D. Pearce, 'Rationalizable strategic behavior and the problem of perfection', *Econometrica*, 52 (1984), 1029–1050
- R.W. Rosenthal, 'A bounded-rationality approach to the study of noncooperative games', *International Journal of Game Theory*, 18 (1989), 273–292
- F. Schuhmacher, 'Proper rationalizability and backward induction', *International Journal of Game Theory*, 28 (1999), 599–615
- T. Tan and S.R.C. Werlang, 'The bayesian foundations of solution concepts of games', *Journal of Economic Theory*, 45 (1988), 370–391