Algorithms for Cautious Reasoning in Games*

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Abstract

We provide comparable algorithms for the Dekel-Fudenberg procedure, iterated admissibility and proper rationalizability by means of the concepts of preference restrictions and likelihood orderings. We apply the algorithms for comparing iterated admissibility and proper rationalizability, and provide a sufficient condition under which iterated admissibility does not rule out properly rationalizable strategies. Finally, we use the algorithms to examine an economically relevant strategic situation, namely a bilateral commitment bargaining game.

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1 Introduction

In non-cooperative game theory, a player is *cautious* if he takes into account all opponents' strategies, also strategies that seem very unlikely to be chosen by the opponent. What outcomes of a strategic game are consistent with common belief of the event that all players are rational and cautious?

A number of contributions, starting with Brandenburger (1992) and Börgers (1994), have shown that the *Dekel-Fudenberg procedure* (Dekel and Fudenberg, 1990), where one round of elimination of *weakly* dominated strategies is followed by iterated elimination of *strongly* dominated strategies, provides a robust answer to this question, in the sense that the eliminated strategies are definitely incompatible with common belief of the event that all players are rational and cautious. The procedure of *iterated admissibility*, which iteratively eliminates all weakly dominated strategies, rules out more strategies. So does the concept of *proper rationalizability.*

Both the Dekel-Fudenberg procedure and iterated admissibility are defined in terms of algorithms. While epistemic foundations for the former was provided relatively quickly (Brandenburger, 1992; Börgers, 1994), half a century elapsed since iterated admissibility was introduced in 1950's before Brandenburger et al. (2008) offered an epistemic foundation for this procedure.

The case of proper rationalizability is different. This concept was defined by Schuhmacher (1999) and Asheim (2001) by means of epistemic conditions. Schuhmacher defines, for every $\varepsilon > 0$, the ε-proper trembling condition, which states that if a player prefers one pure strategy over another, then the probability he assigns to the latter strategy should be at most ε times the probability he assigns to the former. Proper rationalizability is obtained by imposing common belief of the ε-proper trembling condition, and then letting ε tend to zero. Asheim's analysis of 2-player games builds on Blume et al. (1991a,b) in that it uses *lexicographic probability systems* to characterize proper rationalizability. More precisely, Asheim considers the event that players are cautious and *respect the opponent's preferences*, i.e., if the player believes that his opponent prefers one strategy over another, then the player should deem the former infinitely more likely. Any strategy that is a best response under common belief of this event is properly rationalizable.

Schuhmacher (1999) provides an algorithm, *iteratively proper trembling,* which generates for a given $\varepsilon > 0$ the set of mixed strategy profiles that can be chosen under common belief of the ε-proper trembling condition. This procedure does not yield the set of properly rationalizable strategies directly, as we must still let ε go to zero, and see which strategies survive in the limit. Recently, Perea (2008) has provided an algorithm that *directly* computes the set of properly rationalizable strategies in two-player games.

The purpose of the present paper is to present algorithms for the Dekel-Fudenberg procedure and iterated admissibility that build on the key concepts introduced by Perea (2008), thereby making such established procedures comparable to the new algorithm for proper rationalizability. In Section 2, we introduce these key concepts: *preference restrictions* and *likelihood orderings*. In Section 3, we construct algorithms for the Dekel-Fudenberg procedure and iterated admissibility that are comparable with the one for proper rationalizability. In Section 4, we then put these algorithms to use. In particular, we offer examples illuminating the differences between iterated admissibility and proper rationalizability. Moreover, we provide a sufficient condition under which iterated admissibility does not rule out properly rationalizable strategies. Finally, we use the algorithms to examine an economically relevant strategic situation, namely a bilateral commitment bargaining game which has recently been analyzed by Ellingsen and Miettinen (2008). In Section 5 we offer concluding remarks, while an appendix contains all proofs.

2 Preference Restrictions and Likelihood Orderings

Consider a finite *strategic game* $G = (S_1, S_2, u_1, u_2)$ with two players, where the finite set S_i denotes the set of strategies for player i and $u_i : S_1 \times S_2 \to \mathbb{R}$ denotes player i's utility function, for $i \in \{1, 2\}$. As usual, we extend u_i to subjective probability distributions $\lambda_i \in \Delta(A_j)$ over the opponent's strategies, writing $u_i(s_i, \lambda_i)$ for the resulting subjective expected utility.

Each player i's preferences over his own strategies are determined by u_i and a *lexicographic probability system* (LPS) (Blume et al., 1991a) with full support on S_j. An LPS consists of a finite sequence of subjective probability distributions, $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^K)$, where for each $k \in \{1, \ldots, K\}$, $\lambda_i^k \in \Delta(S_j)$. Player i prefers $a_i \in S_i$ to $s_i \in S_i$ if there exists $k \in \{1, \ldots, K\}$ such that (i) $u_i(a_i, \lambda_i^k) > u_i(s_i, \lambda_i^k)$ and (ii) $u_i(a_i, \lambda_i^{k'}) = u_i(s_i, \lambda_i^{k'})$ for all $k' \in \{1, \ldots, k-1\}$. The LPS $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^K)$ has *full support* on S_j if, for all $s_j \in S_j$, there exists $k \in \{1, \ldots, K\}$ such that $\lambda_i^k(s_j) > 0$. Player i deems s_j *infinitely more likely than* s_j' (written $s_j \gg_i s_j'$) if there exists $k \in \{1, \ldots, K\}$ such that (i) $\lambda_i^k(s_j) > 0$ and (ii) $\lambda_i^{k'}(s_j') = 0$ for all

$k' \in \{1, \ldots, k\}$. It follows that $\gg_i$ is an asymmetric and transitive binary relation.

The following two definitions, which are taken from Perea (2008), provide the key concepts for our algorithms.

Definition 1 (Preference restriction) A preference restriction on S_i is a pair (s_i, A_i), where $s_i \in S_i$ and A_i is a nonempty subset of S_i.

The interpretation of a preference restriction (s_i, A_i) is that player i prefers some strategy in A_i to s_i. Let $\mathcal{R}_i^*$ denote the collection of all sets of preference restrictions.

For any set R_i of preference restrictions, define the *choice set* $C_i(R_i)$ as follows:

$$C_i(R_i) := \{s_i \in S_i \mid \nexists A_i \subseteq S_i \text{ with } (s, A_i) \in R_i\}\,.$$

It follows that $C_i(R_i') \cap C_i(R_i'') = C_i(R_i' \cup R_i'')$ for every R_i', $R_i'' \in \mathcal{R}_i^*$. In particular, $C_i(R_i') \supseteq C_i(R_i'')$ whenever $R_i' \subseteq R_i''$.

Definition 2 (Likelihood ordering) A *likelihood ordering* on S_i is an ordered partition $L_i = (L_i^1, L_i^2, \ldots, L_i^K)$ of S_i.

A likelihood ordering $L_i = (L_i^1, L_i^2, \ldots, L_i^K)$ on S_i determines the infinitely-more-likely relation of player j: $s_i \gg_j s_i'$ if and only if $s_i \in L_i^k$ and $s_i' \in L_i^{k'}$ with $k < k'$. Let $\mathcal{L}_i^*$ denote the set of all likelihood orderings on S_i.

For any subset $\mathcal{L}_j$ of likelihood orderings on S_j, let $R_i(\mathcal{L}_j)$ denote the set of preference restrictions *derived* from $\mathcal{L}_j$ in the following manner:

$$\begin{aligned} R_i(\mathcal{L}_j) := \{(s_i, A_i) \in S_i \times 2^{S_i} \mid \forall L_j = (L_j^1, \ldots L_j^K) \in \mathcal{L}_j, \exists k \in \{1, \ldots, K\} \text{ and} \\ \mu_i \in \Delta(A_i) \text{ such that } s_i \text{ is weakly dominated by } \mu_i \text{ on } L_j^1 \cup \cdots \cup L_j^k\}\,. \end{aligned}$$

Here, we say that s_i is *weakly dominated* by μ_i on some subset $A_j \subseteq S_j$ if $u_i(s_i, s_j) \leq u_i(\mu_i, s_j)$ for every $s_j \in A_j$, with strict inequality for some $s_j \in A_j$. It follows that $R_i(\mathcal{L}_j') \cap R_i(\mathcal{L}_j'') = R_i(\mathcal{L}_j' \cup \mathcal{L}_j'')$ for every $\mathcal{L}_j'$, $\mathcal{L}_j'' \in \mathcal{L}_i^*$. In particular, $R_i(\mathcal{L}_j') \supseteq R_i(\mathcal{L}_j'')$ whenever $\mathcal{L}_j' \subseteq \mathcal{L}_j''$.

Likelihood-orderings can be related to the ordinary *belief* operator as well as the *assumption* operator, as proposed by Brandenburger et al. (2008) (and discussed by Asheim and Søvik, 2005, Section 6).

Definition 3 (Believing an event) For a given subset $A_i \subseteq S_i$, we say that the likelihood ordering L_i *believes* A_i if, for every $s_i \in S_i \backslash A_i$, $a_i \gg_j s_i$ for some $a_i \in S_i$.

Definition 4 (Assuming an event) For a given subset $A_i \subseteq S_i$, we say that the likelihood ordering L_i *assumes* A_i if, for every $s_i \in S_i \backslash A_i$, $a_i \gg_j s_i$ for every $a_i \in A_i$.

So, if L_i assumes a non-empty event A_i it also believes the event A_i, but not vice versa. Likelihood-orderings can also be related to *respect of preferences* as introduced by Blume et al. (1991b).

Definition 5 (Respecting preferences) For a given subset $R_i \subseteq \mathcal{R}_i^*$ of preference restrictions, we say that the likelihood ordering L_i *respects* R_i if, for every $(s_i, A_i) \in R_i$, $a_i \gg_j s_i$ for some $a_i \in A_i$.

Hence, if L_i respects the set R_i of preference restrictions, it also believes the event $C_i(R_i)$, but not vice versa.

Let $\mathcal{L}_i^b(R_i)$ denote the set of likelihood orderings that believe player i's rationality when i's preferences satisfy the set R_i of preference restrictions:

$$\mathcal{L}_i^b(R_i) := \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } C_i(R_i)\} .$$

Let $\mathcal{L}_i^a(R_i)$ denote the set of likelihood orderings that assume player i's rationality when i's preferences satisfy the set R_i of preference restrictions:

$$\mathcal{L}_i^a(R_i) := \{L_i \in \mathcal{L}_i^* \mid L_i \text{ assumes } C_i(R_i)\} .$$

Finally, let $\mathcal{L}_i^r(R_i)$ denote the set of likelihood orderings that *respect* player i's preferences when i's preferences satisfy the set R_i of preference restrictions:

$$\mathcal{L}_i^r(R_i) := \{L_i \in \mathcal{L}_i^* \mid L_i \text{ respects } R_i\} .$$

It follows from the observations that assumption implies belief, but not vice versa, and respect of preferences implies belief of rationality, but not versa, that

$$\mathcal{L}_i^b(R_i) \supseteq \mathcal{L}_i^a(R_i) \cup \mathcal{L}_i^r(R_i)$$

for every $R_i \in \mathcal{R}_i^*$ with $C_i(R_i) \neq \emptyset$. Since the belief operator satisfies conjunction and monotonicity, the properties of the choice correspondence $C_i(\cdot)$ imply that

$$\mathcal{L}_i^b(R_i') \cap \mathcal{L}_i^b(R_i'') = \mathcal{L}_i^b(R_i' \cup R_i'') .$$

for every R_i', $R_i'' \in \mathcal{R}_i^*$. However, since the assumption operator satisfies conjunction but not monotonicity, it holds for every R_i', $R_i'' \in \mathcal{R}_i^*$ that

$$\mathcal{L}_i^a(R_i') \cap \mathcal{L}_i^a(R_i'') \subseteq \mathcal{L}_i^a(R_i' \cup R_i'') ,$$

while the inverse inclusion need not hold. Finally, Definition 5 implies that

$$\mathcal{L}_i^r(R_i') \cap \mathcal{L}_i^r(R_i'') = \mathcal{L}_i^r(R_i' \cup R_i'') \,.$$

In particular, $\mathcal{L}_i^b(R_i') \supseteq \mathcal{L}_i^b(R_i'')$ and $\mathcal{L}_i^r(R_i') \supseteq \mathcal{L}_i^r(R_i'')$ whenever $R_i' \subseteq R_i''$. This conclusion need not hold for $\mathcal{L}_i^a(\cdot)$ since a likelihood ordering L_i may assume A_i' but not A_i'' even though $A_i' \subset A_i''$. Hence, we may have $\mathcal{L}_i^a(R_i') \nsubseteq \mathcal{L}_i^a(R_i'')$ and $\mathcal{L}_i^a(R_i') \nsupseteq \mathcal{L}_i^a(R_i'')$ even though $R_i' \subset R_i''$.

3 Algorithms

In this section we provide comparable algorithms for the Dekel-Fudenberg procedure, iterated admissibility and proper rationalizability.

3.1 An algorithm for the Dekel-Fudenberg procedure

We first consider the *Dekel-Fudenberg procedure* (Dekel and Fudenberg, 1990), which is the procedure where one round of maximal elimination of weakly dominated strategies is followed by iterated maximal elimination of strictly dominated strategies. Following Brandenburger (1992), strategies surviving the Dekel-Fudenberg procedure are referred to as *permissible*.

Consider the following algorithm, which iteratedly increases the set of preference restrictions for both players:

Ini For both players i, let $R_i^0 = \emptyset$.

DF For every $n \geq 1$, and both players i, let $R_i^n = R_i(\mathcal{L}_j^b(R_j^{n-1}))$.

From the properties of $\mathcal{L}_i^b(\cdot)$ and $R_i(\cdot)$, it follows that **Ini** and **DF** determines, for each player, a non-decreasing sequence of sets of preference restrictions and a non-increasing sequence of sets of likelihood orderings (where non-decreasing and non-increasing are defined w.r.t. set inclusion). As a consequence, the sequence $C_i(R_i^n)$ of choice sets is non-increasing. Since the set of preference restrictions is finite, the algorithm converges after a finite number of rounds.

For both players i, let $R_i^\infty := \bigcup_{n=1}^\infty R_i^n$ be the limiting set of preference restrictions produced by the algorithm defined by **Ini** and **DF**.

Proposition 1 *Let G be a finite two-player strategic game. Then, for both players i, a strategy s_i is permissible if and only if $s_i \in C_i(R_i^\infty)$.*

Proof. See the appendix. ■

3.2 An algorithm for iterated admissibility

Iterated admissibility is the procedure of iterated maximal elimination of weakly dominated strategies.

Consider the following algorithm:

Ini For both players i, let $R_i^0 = \emptyset$.

IA For every $n \geq 1$, and both players i, let

$$R_i^n = R_i\big(\mathcal{L}_j^a(R_j^0) \cap \mathcal{L}_j^a(R_j^1) \cap \cdots \cap \mathcal{L}_j^a(R_j^{n-1})\big) .$$

From the properties of $R_i(\cdot)$, it follows that **Ini** and **IA** determines, for each player, a non-decreasing sequence of sets of preference restrictions and a non-increasing sequence $\mathcal{L}_j^a(R_j^0) \cap \mathcal{L}_j^a(R_j^1) \cap \cdots \cap \mathcal{L}_j^a(R_j^n)$ of sets of likelihood orderings. As a consequence, the sequence $C_i(R_i^n)$ of choice sets is non-increasing. Since the set of preference restrictions is finite, the algorithm converges after a finite number of rounds.

For both players i, let $R_i^\infty := \bigcup_{n=1}^\infty R_i^n$ be the limiting set of preference restrictions produced by the algorithm defined by **Ini** and **IA**.

Proposition 2 *Let G be a finite two-player strategic game. Then, for both players i, a strategy s_i survives iterated admissibility if and only if $s_i \in C_i(R_i^\infty)$.*

Proof. See the appendix. ■

Proposition 2 echoes Brandenburger et al.'s (2008, Theorem 9.1) epistemic characterization of iterated admissibility (see also the observation that Stahl, 1995, makes in his theorem), by pointing out that iterated admissibility corresponds to likelihood orderings where strategies eliminated in a later round are deemed infinitely more likely than strategies eliminated in an earlier round, and surviving strategies are deemed infinitely more likely than strategies eliminated in some round. Here we let these likelihood orderings interplay with sets of preference restrictions, thereby allowing comparison with the algorithm for proper rationalizability, presented next.

3.3 An algorithm for proper rationalizability

We finally consider *proper rationalizability*, a concept defined by Schuhmacher (1999) and characterized by Asheim (2001). We refer to these references for details.

Consider the following algorithm:

Ini For both players i, let $R_i^0 = \emptyset$.

PR For every $n \geq 1$, and both players i, let $R_i^n = R_i(\mathcal{L}_j^r(R_j^{n-1}))$.

From the properties of $\mathcal{L}_i^r(\cdot)$ and $R_i(\cdot)$, it follows that **Ini** and **PR** determines, for each player, a non-decreasing sequence of sets of preference restrictions and a non-increasing sequence of sets of likelihood orderings. Since the set of preference restrictions is finite, the algorithm converges after a finite number of rounds.

For both players i, let $R_i^\infty := \bigcup_{n=1}^\infty R_i^n$ be the limiting set of preference restrictions produced by the algorithm defined by **Ini** and **PR**.

Proposition 3 *Let G be a finite two-player strategic game. Then, for both players i, a strategy s_i is properly rationalizable if and only if $s_i \in C_i(R_i^\infty)$.*

Proof. Perea (2008). ■

4 Applying the algorithms

In this section we put the algorithms to work. In the first subsection we present three examples illustrating how the sequences of preference restrictions that the algorithms give rise to shed light on differences between iterated admissibility and proper rationalizability. In particular, in the first example, the set of strategies surviving iterated admissibility is a strict subset of the set of properly rationalizable strategies, while the sequences of preference restrictions for iterated admissibility and proper rationalizability coincide in the latter two examples.

In the second subsection we build on insights conveyed by the examples and provide through Proposition 4 a sufficient condition ensuring that any properly rationalizable strategy survives iterated admissibility. In particular, since proper equilibrium always exists and any strategy being used with positive probability in a proper equilibrium is properly rationalizable, we reach the following conclusion: If a game—for which iterated admissibility leads to a unique strategy for each player—satisfies the sufficient condition of Proposition 4, then the surviving strategies are the unique properly rationalizable strategies and the corresponding strategy profile is the unique proper equilibrium.

In the third subsection we consider a recent contribution on commitment bargaining (Ellingsen and Miettinen, 2008) and use the algorithm of Section 3.3 to show how proper rationalizability yields the results they seek, while other procedures do not.

4.1 Examples

We now illustrate our algorithms by means of three examples. Before doing so, we introduce the following piece of notation: For a given set R_i of preference restrictions on S_i, define the monotonic cover of R_i by

$$mcR_i := \{(s_i, A_i) \mid \exists \hat{A}_i \subseteq A_i \text{ with } (s_i, \hat{A}_i) \in R_i\} .$$

Every set R_i^n of preference restrictions produced by each algorithm on the way to R_i^∞ can clearly be written as the monotonic cover of some smaller set.

[Figure 1 about here.]

In G_1, illustrated in Figure 1 (and discussed by Asheim and Dufwenberg, 2003), iterated admissibility works by eliminating D, R, and M, leading to (U, L), while the concept of proper rationalizability rules out just D. In the first round, the only restriction imposed by both iterated admissibility and proper rationalizability is that U is preferred to D and thus, (s_1, A_1) is a preference restriction for 1 if and only if $s_1 = D$ and $A_1 \ni U$ (which in the notation just introduced is written $R_1^1 = mc\{(D, \{U\})\}$). In the algorithm of proper rationalizability, this means that the likelihood ordering over player 1's strategies must satisfy that U is infinitely more likely than D. Since this does not imply anything about the relative likelihood of M and D, which is what the preferences of player 2 depend on, no preference restriction is imposed on 2. Thus the algorithm converges after one round.

In contrast, since $C_1(mc\{(D, \{U\})\}) = \{U, M\}$, a likelihood ordering assumes $C_1(mc\{(D, \{U\})\})$ if *each* of U and M is infinitely more likely than D. This in turn means that L is preferred to R and U is preferred to M in the algorithm of preference restrictions that characterizes iterated admissibility (cf. Section 3.2), with $(\{U\}, \{M\}, \{D\})$ and $(\{L\}, \{R\})$ as the corresponding likelihood orderings. The likelihood ordering, $(\{L\}, \{R\})$, for player 2 entails that player 1 deems L infinitely more likely that R and therefore prefers D to M (and, of course, U to D since the former weakly dominates the latter). However, this means that the likelihood ordering, $(\{U\}, \{M\}, \{D\})$, for player 1 determined by the algorithm characterizing iterated admissibility does not respect the preferences of player 1 that the same algorithm give rise to.

[Figure 2 about here.]

Compare G_1 to G_2, which is the game illustrated in Figure 2. In G_2, the algorithms of iterated admissibility and proper rationalizability coincide in terms of the sets of preference restrictions. In the first round, the only restriction imposed by both iterated admissibility and proper rationalizability is that U is preferred to D; i.e., $R_1^1 = mc\{(D, \{U\})\}$. Even though the set of likelihood orderings that assumes $C_1(mc\{(D, \{U\})\})$ is a strict subset of the set of likelihood orderings that respects $mc\{(D, \{U\})\}$ (since only the former requires that M must be deemed infinitely more likely than D), every member of each set deems U infinitely more likely than D. This is sufficient to conclude L is preferred to R and U is preferred to M in the algorithms of iterated admissibility and proper rationalizability.

A key observation for game G_2 is that U weakly dominates D, and that L weakly dominates R on both $\{U\}$ (which is the strategy used to eliminate D in the first round of iterated admissibility) and $\{U, M\}$ (which is the set of strategies for player 1 surviving the first round of iterated admissibility). The same kind of observation can be made for the centipede game, which we turn to next.

[Figure 3 about here.]

In the four-legged centipede game illustrated in Figure 3 it is also the case that the algorithms of iterated admissibility and proper rationalizability coincide in terms of the sets of preference restrictions. In the first round, the only restriction imposed by both iterated admissibility and proper rationalizability is that *fd* is preferred to *ff*; i.e., $R_2^1 = mc\{(\mathit{ff}, \{\mathit{fd}\})\}$. Even though the set of likelihood orderings that assumes $C_2(mc\{(\mathit{ff}, \{\mathit{fd}\})\})$ is a strict subset of the set of likelihood orderings that respects $mc\{(\mathit{ff}, \{\mathit{fd}\})\}$ (since only the former requires that d must be deemed infinitely more likely than *ff*), every member of each set deems *fd* infinitely more likely than *ff*. This is sufficient to conclude *FD* is preferred to *FF*. Even though the set of likelihood orderings that assumes $C_1(mc\{(FF, \{FD\})\})$ is a strict subset of the set of likelihood orderings that respects $mc\{(FF, \{FD\})\}$ (since only the former requires that D must be deemed infinitely more likely than FF), every member of each set deems *FD* infinitely more likely than *FF*. This is sufficient to conclude d is preferred to *fd* and D is preferred to *FD*.

Note that in the second round, *FD* weakly dominates *FF* on both $\{fd\}$ (which is the strategy used to eliminate *ff* in the first round of iterated admissibility) and $\{d, fd\}$ (which is the set of strategies for player 2 surviving the first round of iterated admissibility). Likewise, in the third round, d weakly dominates *fd* and *ff* on both $\{FD\}$ (which is the strategy used to eliminate *FF* in the second round of iterated admissibility) and $\{D, FD\}$ (which is the set of strategies for player 1 surviving the second round of iterated admissibility). Similar conclusions hold for any centipede game independent of size and illustrates how both iterated admissibility and proper rationalizability correspond to the procedure of backward induction in such games.[1]

4.2 A sufficient condition

The following proposition presents a sufficient condition under which iterated admissibilty does not rule out properly rationalizable strategies.

Proposition 4 *Consider a finite two-player strategic game G where the procedure of iterated admissibility leads to the sequence $\langle S_1^n, S_2^n \rangle_{n=0}^{\infty}$ of surviving strategy sets. Suppose that there exists a sequence $\langle A_1^n, A_2^n \rangle_{n=0}^{\infty}$ of strategy sets satisfying, for both players i, $A_i^0 = S_i$ and for each $n \in \mathbb{N}$,*

- $A_i^n \subseteq S_i^n$,
- *if $S_i^n \neq S_i^{n-1}$, then, for every $s_i \in S_i \backslash S_i^n$, s_i is weakly dominated by every $a_i \in A_i^n$ on either (A_j^{n-1} and S_j^{n-1}) or S_j,*
- *if $S_i^n = S_i^{n-1}$, then $A_i^n = A_i^{n-1}$.*

Then, for both players i, if s_i is properly rationalizable, then $s_i \in \bigcap_{n=1}^{\infty} S_i^n$.

Proof. See the appendix. ■

[1]For finite perfect information games without relevant payoff ties, proper rationalizability leads to the unique profile of backward induction *strategies* (Schuhmacher, 1999; Asheim, 2001), and iterated admissibility leads to the backward induction *outcome* (see Battigalli, 1997, pp. 52–53, for relevant references). While the algorithms of Sections 3.2 and 3.3 correspond to the backward induction *procedure* in the subclass of centipede games, this does not hold for the whole class of finite perfect information games without relevant payoff ties.

Both G_2 of Figure 2 and G_3 of Figure 3 can be used to illustrate Proposition 4. In G_2, the procedure of iterated admissibility yields the following sequence of strategy sets: $S_1^1 = S_1^2 = \{U, M\}$ and $S_1^n = \{U\}$ for $n \geq 3$, and $S_2^1 = \{L, R\}$ and $S_2^n = \{L\}$ for $n \geq 2$. Choose $A_1^n = \{U\}$ for $n \geq 1$, and $A_2^1 = \{L, R\}$ and $A_2^n = \{L\}$ for $n \geq 2$. It is straightforward to check that the conditions of Proposition 4 are satisfied; in particular, L weakly dominates R on both $A_1^1 = \{U\}$ and $S_1^1 = \{U, M\}$, and U weakly dominates M on $A_2^2 = S_2^2 = \{L\}$, and weakly dominates D on S_2.

In G_3, the procedure of iterated admissibility yields the following sequence of strategy sets: $S_1^1 = \{D, FD, FF\}$, $S_1^2 = S_1^3 = \{D, FD\}$ and $S_1^n = \{D\}$ for $n \geq 4$, and $S_2^1 = S_2^2 = \{d, fd\}$ and $S_1^n = \{d\}$ for $n \geq 3$. Choose $A_1^1 = \{D, FD, FF\}$, $A_1^2 = A_1^3 = \{FD\}$ and $A_1^n = \{D\}$ for $n \geq 4$, and $A_2^1 = A_2^2 = \{fd\}$ and $A_2^n = \{d\}$ for $n \geq 3$. Again, it is straightforward to check that the conditions of Proposition 4 are satisfied; in particular, FD weakly dominates FF on both $A_2^1 = \{fd\}$ and $S_2^1 = \{d, fd\}$, d weakly dominates both fd and ff on both $A_1^2 = \{FD\}$ and $S_1^2 = \{D, FD\}$, and D weakly dominates both FD and FF on $A_2^3 = S_2^3 = \{d\}$.

4.3 Commitment bargaining

The algorithms of Section 3 can be applied for the purpose of analyzing economically significant models, independently of whether the sufficient condition of Proposition 4 is satisfied. In particular, they can be used for comparing iterated admissibility to properly rationalizability strategies in specific strategic situations. In this subsection we consider a model of bilateral commitment bargaining due to Crawford (1982). Ellingsen and Miettinen (2008) extends Crawford's (1982) analysis by considering iterated admissibility and refinements of Nash equilibrium. Here we show how some of the results of Ellingsen and Miettinen (2008), in particular Lemma 2 and Proposition 2, can be obtained by using proper rationalizability instead of iterated admissibility. We also believe there is a mistake in their Lemma 2, but we will come back to this later.

In order to turn their strategic situation where two players bargain over real numbered fractions of a surplus of size 1 into a *finite* one-stage game with simultaneous moves, we introduce a smallest money unit g. We measure all variables in terms of numbers of the smallest money unit, and assume that k units of the smallest money unit equals the total surplus (i.e., $k \cdot g = 1$). Hence, players 1 and 2 bargain over a surplus of size k. Each player i chooses, simultaneously with the other, either to commit to some demand $s_i \in \{0, 1, \ldots, k\}$ or to wait and remain uncommitted. Let w denote the waiting strategy. Hence the strategy set of each player i is $S_i = \{0, 1, \ldots, k\} \cup \{w\}$. If both players choose w, then each player i receives $\beta_i > 0$, where $\beta_1 + \beta_2 = k$. If only one player i makes a commitment s_i, then i receives s_i and the other player receives $k - s_i$. If both players make commitments, then each player i receives $x_i(s_i, s_j) \in \{s_i, s_i + 1, \ldots, k - s_j\}$, with $x_1(s_1, s_2) + x_2(s_1, s_2) \leq k$, if $s_1 + s_2 \leq k$ and nothing otherwise.

The payoff function $u_i(s_i, s_j)$ of each player i can be summarized as follows:

$$u_i(s_i, s_j) \;=\; \begin{cases} x_i(s_i, s_j) & \text{if } s_i + s_j \leq k\,, \\ 0 & \text{if } s_i + s_j > k\,, \\ s_i & \text{if } s_i \neq w \text{ and } s_j = w\,, \\ k - s_j & \text{if } s_i = w \text{ and } s_j \neq w\,, \\ \beta_i & \text{if } s_i = w = s_j\,. \end{cases}$$

Ellingsen and Miettinen (2008) claim through their Lemma 2 that, for each player i, iterated admissibility leads to the elimination of $0, 1, \ldots, \beta_i$ in the first round, and $\beta_i{+}1, \beta{+}2, \ldots, k{-}1$ in the second round, leaving k and w as the surviving strategies. Actually, with only k and w as the surviving strategies, w is eliminated in the third round, since choosing k yields player i a payoff of 0 if the opponent also chooses k and k if the opponent chooses w, while choosing w yields player i a payoff of 0 if the opponent chooses k and β_i ($< k$) if the opponent also chooses w. Hence, the correct statement of Ellingsen and Miettinen's (2008) Lemma 2 is that only k is iteratively weakly undominated.

Ellingsen and Miettinen (2008) use Lemma 2 in their subsequent Proposition 2 to focus on Nash equilibria involving only the strategies k and w (including asymmetric equilibria where one commits to the entire surplus and the other waits), as opposed to the plethora of unrefined Nash equilibria that this game gives rise to (cf. Crawford, 1982). Their Proposition 2 states that only the two asymmetric equilibria along with the symmetric equilibrium where both claim the entire surplus are consistent with two rounds of elimination of weakly dominated strategies. This statement is correct, but it begs the question: *why stop with two rounds of weak elimination?* As the following proposition shows, proper rationalizability provides a reason for considering only the strategies k and w.

Proposition 5 *Consider the finite version of Crawford's (1982) and Ellingsen and Miettinen's (2008) bilateral commitment bargaining game. The properly rationaliz-*

able strategies for each player is to commit to the whole surplus, i.e., to choose the strategy k, or to wait, i.e., to choose the strategy w.

Proof. See the appendix. ■

The proof of Proposition 5 consists of two parts. The one part uses the algorithm of Section 3.3 to show that no strategy but k and w can be properly rationalizable. The other part uses the result of Asheim (2001, Proposition 2)—that any strategy being used with positive probability in a proper equilibrium is properly rationalizable—to show that k and w are properly rationalizable. In particular, the asymmetric equilibria where one player commits to the entire surplus and the other waits are proper. In addition, there is a proper equilibrium where both player choose k with probability 1.[2] It can be shown that in any perfect equilibrium (and thus, in any proper equilibrium), at least one player chooses k with probability 1, at most one player chooses w with positive probability, and no other strategy is assigned positive probability. Thus, the concept of perfect equilibrium focuses precisely on the equilibria highlighted in Ellingsen and Miettinen's (2008) Proposition 2.

Ellingsen and Miettinen (2008) also consider a variant of Crawford's (1982) bilateral commitment bargaining game. In their Proposition 4 they show that only k survives iterated admissibility if commitments are uncertain. Actually, the iterations involve one round of weak elimination, followed by two rounds of strict elimination. Hence, only k survives the Dekel-Fudenberg procedure, and it follows from the algorithms of Sections 3.1 and 3.3 that only k is properly rationalizable (and thus, (k, k) is the only proper equilibrium). In their Propositions 1 and 3 they consider costly commitments. In this case, it can be shown that every strategy surviving iterated elimination of *strictly* dominated strategies is properly rationalizable. Hence in all variants considered by Ellingsen and Miettinen (2008), proper rationalizability yields the results they seek, while other procedures do not.

5 Concluding remarks

In our opinion, proper rationalizability is an attractive concept which is based on appealing epistemic conditions. However, up to now, its applicability has been hampered by the lack of an algorithm leading directly to the properly rationalizable strategies. With Perea's (2008) algorithm, this roadblock has been removed.

[2]This equilibrium involves likelihood orderings where $k-1$ and w are at the second level.

In this paper we have compared proper rationalizability to the Dekel-Fudenberg procedure and iterated admissibility by presenting comparable algorithms for the two latter concepts. Through the bilateral commitment bargaining game due to Crawford (1982) and Ellingsen and Miettinen (2008) we have illustrated the usefulness of proper rationalizability in economic applications.

A Proofs

In order to prove Proposition 1, we need the following lemma.

Lemma 1 *Let $s_i \in S_i$, $D_i \subseteq S_i$ and $D_j \subseteq S_j$. Then, s_i is strictly dominated by some $\mu_i \in \Delta(D_i)$ on D_j if and only for every $(\emptyset \neq)\ E_j \subseteq D_j$ strategy s_i is weakly dominated by some $\tilde{\mu}_i \in \Delta(D_i)$ on E_j.*

Proof. *Only if.* If there exists $\mu_i \in \Delta(D_i)$ such that μ_i strictly dominates s_i on D_j, then, for every $(\emptyset \neq)\ E_j \subseteq D_j$, $\mu_i \in \Delta(D_i)$ weakly dominates s_i on E_j.

If. Suppose there does not exist $\mu_i \in \Delta(D_i)$ such that μ_i strictly dominates s_i on D_j. Hence, by Pearce (1984, Lemma 3), there exists $\lambda_i \in \Delta(D_j)$ such that $u(s_i, \lambda_i) \geq u(s_i', \lambda_i)$ for all $s_i' \in D_i$. Then, by Pearce (1984, Lemma 4), there does not exist $\tilde{\mu}_i \in \Delta(D_i)$ such that $\tilde{\mu}_i$ weakly dominates s_i on $E_j := \text{supp}\lambda_i \subseteq D_j$. ■

Proof of Proposition 1. The Dekel-Fudenberg procedure is given by the following sequence of strategy subsets:

(i) For each player i, let $S_i^0 = S_i$.

(ii) For each player i, let $S_i^1 = \{s_i \in S_i \mid s_i \text{ not weakly dominated on } S_j\}$.

(iii) For every $n \geq 2$, and each player i, let

$$S_i^n = \{s_i \in S_i^{n-1} \mid s_i \text{ not strictly dominated on } S_j^{n-1}\}.$$

We show, by induction on n, that $C_i(R_i^n) = S_i^n$ for each player i and all n.

Part (i). For $n = 0$, we have that

$$C_i(R_i^0) = C_i(\emptyset) = S_i = S_i^0$$

for each player i.

Part (ii). For $n = 1$, we have that $R_i^1 = R_i(\mathcal{L}_j^b(R_j^0)) = R_i(\mathcal{L}_j^b(\emptyset))$. By definition,

$$\begin{aligned}\mathcal{L}_j^b(\emptyset) &= \{L_j \in \mathcal{L}_j^* \mid L_j \text{ believes } C_j(\emptyset)\} \\ &= \{L_j \in \mathcal{L}_j^* \mid L_j \text{ believes } S_j\} = \mathcal{L}_j^*.\end{aligned}$$

Hence,

$$R_i^1 = \{(s_i, A_i) \mid \exists \mu_i \in \Delta(A_i) \text{ that weakly dominates } s_i \text{ on } S_j\}.$$

Therefore,

$$C_i(R_i^1) = \{s_i \in S_i \mid s_i \text{ not weakly dominated on } S_j\} = S_i^1.$$

Part (iii). Now, let $n \geq 2$, and assume that for each player i, $C_i(R_i^{n-1}) = S_i^{n-1}$. We show that, for each player i, $C_i(R_i^n) = S_i^n$.

Fix a player i. By definition, $R_i^n = R_i(\mathcal{L}_j^b(R_j^{n-1}))$. We have that

$$\begin{aligned}\mathcal{L}_j^b(R_j^{n-1}) &= \{L_j \in \mathcal{L}_j^* \mid L_j \text{ believes } C_j(R_j^{n-1})\} \\ &= \{L_j \in \mathcal{L}_j^* \mid L_j^1 \subseteq C_j(R_j^{n-1})\} \\ &= \{L_j \in \mathcal{L}_j \mid L_j^1 \subseteq S_j^{n-1}\},\end{aligned}$$

by our induction assumption. But then,

$$\begin{aligned}R_i^n = \{(s_i, A_i) \mid &\text{ for every } L_j^1 \subseteq S_j^{n-1} \text{ there is } \mu_i \in \Delta(A_i) \text{ that} \\ &\text{weakly dominates } s_i \text{ on } L_j^1 \text{ or on } S_j\}.\end{aligned}$$

Consider the strategies s_i where for every $L_j^1 \subseteq S_j^{n-1}$ there is $\mu_i \in \Delta(A_i)$ such that μ_i weakly dominates s_i on L_j^1. By Lemma 1, we know that these are exactly the strategies s_i that are strictly dominated by some $\mu_i \in \Delta(A_i)$ on S_j^{n-1}.

Hence, we may conclude that

$$\begin{aligned}R_i^n = \{(s_i, A_i) \mid &\text{ there is } \mu_i \in \Delta(A_i) \text{ that} \\ &\text{strictly dominates } s_i \text{ on } S_j^{n-1}, \text{ or weakly dominates } s_i \text{ on } S_j\}.\end{aligned}$$

Hence,

$$\begin{aligned}C_i(R_i^n) = \{s_i \in S_i \mid &\ s_i \text{ not strictly dominated on } S_j^{n-1}, \\ &\text{nor weakly dominated on } S_j\} = S_i^n,\end{aligned}$$

which completes the proof. ■

Proof of Proposition 2. Iterated admissibility is given by the following sequence of strategy subsets:

(i) For each player i, let $S_i^0 = S_i$.

(ii) For every $n \geq 1$, and each player i, let

$$S_i^n = \{s_i \in S_i^{n-1} \mid s_i \text{ not weakly dominated on } S_j^{n-1}\}.$$

We show, by induction on n, that $C_i(R_i^n) = S_i^n$ for each player i and all n.

Part (i). For $n = 0$, we have that

$$C_i(R_i^0) = C_i(\emptyset) = S_i = S_i^0$$

for each player i.

Part (ii). Let $n \geq 1$, and assume that, for each player i, $C_i(R_i^k) = S_i^k$ for all $k < n$. We show that, for each player i, $C_i(R_i^n) = S_i^n$.

Fix a player i. By definition, we have that

$$R_i^n = R_i\big(\mathcal{L}_j^a(R_j^0) \cap \mathcal{L}_j^a(R_j^1) \cap \cdots \cap \mathcal{L}_j^a(R_j^{n-1})\big)\,.$$

By the induction assumption, we know that $C_j(R_j^k) = S_j^k$ for all $k < n$, and hence

$$\begin{aligned} \mathcal{L}_j^a(R_j^k) &= \{L_j \in \mathcal{L}_j^* \mid L_j \text{ assumes } C_j(R_j^k)\} \\ &= \{L_j \in \mathcal{L}_j^* \mid L_j \text{ assumes } S_j^k\} \end{aligned}$$

for all $k < n$. This implies that $\mathcal{L}_j^a(R_j^0) \cap \mathcal{L}_j^a(R_j^1) \cap \cdots \cap \mathcal{L}_j^a(R_j^{n-1})$ is equal to

$$\{L_j \in \mathcal{L}_j^* \mid L_j \text{ assumes } S_j^k \text{ for all } k < n\}\,.$$

Since $R_i^n = R_i\big(\mathcal{L}_j^a(R_j^0) \cap \mathcal{L}_j^a(R_j^1) \cap \cdots \cap \mathcal{L}_j^a(R_j^{n-1})\big)$, it follows that R_i^n contains exactly those preference restrictions (s_i, A_i) such that s_i is weakly dominated by some $\mu_i \in \Delta(A_i)$ on some S_j^k with $k < n$. Hence,

$$C_i(R_i^n) = \{s_i \in S_i \mid s_i \text{ not weakly dominated on any } S_j^k \text{ with } k < n\} = S_i^n\,,$$

which completes the proof. ■

Proof of Proposition 4. Let $\langle R_1^n, R_2^n \rangle_{n=1}^{\infty}$ be the sequence of preference restrictions according to the algorithm of proper rationalizability (cf. Section 3.3). It is sufficient to show, under the assumptions of the proposition, that for both player i and each n, it holds that, for every $s_i \in S_i \backslash S_i^n$, $(s_i, \{a_i\}) \in R_i^n$ for every $a_i \in A_i^n$. In this case, namely, every properly rationalizable strategy must be in $\bigcap_{n=1}^{\infty} S_i^n$. We show by induction that the statement above is true.

Part (i). For $n = 0$, we have that $S_i^0 = S_i$, so that there is no $s_i \in S_i \backslash S_i^n$ and the statement is trivially true.

Part (ii). Let $n \geq 1$, and assume that, for each player i and each $m \in \{1, \ldots, n-1\}$, it holds that, for every $s_i \in S_i \backslash S_i^m$, $(s_i, \{a_i\}) \in R_i^m$ for every $a_i \in A_i^m$.

Fix a player i. We first make the observation that, for each $m \in \{1, \ldots, n-1\}$, every $L_j = (L_j^1, \ldots, L_j^K) \in \mathcal{L}_j^r(R_j^m)$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $A_j^m \subseteq L_j^1 \cup \cdots \cup L_j^k \subseteq S_j^m$. This is true by the full support assumption if $S_j^m = S_j$ (and thus $A_j^m = S_j$). Assume now that $S_j^m \neq S_j$. If L_j respects R_j^m, then for every $s_j \in S_j \backslash S_j^m$, $a_j \gg_i s_j$ for every $a_j \in A_j^m$, and the observation follows also in this case.

If $S_i^n = S_i$, then the statement is trivially true also for n.

If $S_i^n \neq S_i$, let $(1 \leq)\ n' \leq n$ satisfy $S_i^n = S_i^{n'} \neq S_i^{n'-1}$. By a premise of the proposition, for every $s_i \in S_i \backslash S_i^{n'}$, s_i is weakly dominated by every $a_i \in A_i^{n'}$ on either ($A_j^{n'-1}$ and $S_j^{n'-1}$) or S_j. If s_i is weakly dominated by a_i on $A_j^{n'-1}$ and $S_j^{n'-1}$, then s_i is weakly dominated by a_i on each strategy set D_j satisfying $A_j^{n'-1} \subseteq D_j \subseteq S_j^{n'-1}$. By the observation that every $L_j = (L_j^1, \ldots, L_j^K) \in \mathcal{L}_j^r(R_j^{n'-1})$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $A_j^{n'-1} \subseteq L_j^1 \cup \cdots \cup L_j^k \subseteq S_j^{n'-1}$ it follows that $(s_i, \{a_i\}) \in R_i^{n'} = R_i(\mathcal{L}_j^r(R_j^{n'-1}))$. If s_i is weakly dominated by a_i on S_j, then by the full support assumption, $(s_i, \{a_i\}) \in R_i^1 = R_i(\mathcal{L}_j^*)$. Hence, since the sequence of sets of preference restrictions is non-decreasing, for every $s_i \in S_i \backslash S_i^n$, $(s_i, \{a_i\}) \in R_i^n$ for every $a_i \in A_i^n$. ■

Proof of Proposition 5. The proof is divided into two parts. In part (i) we show that the strategies in $S_i \backslash (\{k\} \cup \{w\})$ are *not* properly rationalizable. In part (ii) we show that k and w *are* properly rationalizable.

Part (i). Let $\langle R_1^n, R_2^n \rangle_{n=1}^{\infty}$ be the sequence of preference restrictions for the finite version of Crawford's (1982) and Ellingsen and Miettinen's (2008) bilateral commitment bargaining game, according to the algorithm of proper rationalizability (cf. Section 3.3). In order to show that the strategies in $S_i \backslash (\{k\} \cup \{w\}) = \{0, 1, \ldots, k-1\}$ are not properly rationalizable, it is sufficient to show that for each player i, it holds that (a) for every $s_i \in \{0, 1, \ldots, \beta_i\}$, $(s_i, \{w\}) \in R_i^1$, and (b) for every $s_i \in \{\beta_i + 1, \beta_i + 2, \ldots, k-1\}$, $(s_i, \{k\}) \in R_i^2$, keeping in mind that the sequence of sets of preference restrictions is non-decreasing.

Result (a) follows from the fact that, for each player i and for every $s_i \in \{0, 1, \ldots, \beta_i\}$, w weakly dominates s_i on S_j. Hence, for each player i and for every

$s_i \in \{0, 1, \ldots, \beta_i\}$, $(s_i, \{w\}) \in R_i^1 = R_i(\mathcal{L}_j^*)$. This result implies that, for each player j, every $L_j = (L_j^1, \ldots, L_j^K) \in \mathcal{L}_j^r(R_j^1)$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $\{w\} \subseteq L_j^1 \cup \cdots \cup L_j^k \subseteq \{\beta_j + 1, \beta_j + 2, \ldots, k\} \cup \{w\}$. Result (b) follows from the fact that, for each player i and for every $s_i \in \{\beta_i + 1, \beta_i + 2, \ldots, k - 1\}$, k weakly dominates s_i on each strategy set D_j satisfying $\{w\} \subseteq D_j \subseteq \{\beta_j + 1, \beta_j + 2, \ldots, k\} \cup \{w\}$. Hence, for each player i and for every $s_i \in \{\beta_i + 1, \beta_i + 2, \ldots, k - 1\}$, $(s_i, \{k\}) \in R_i^2 = R_i(\mathcal{L}_j^r(R_j^1))$.

Part (ii). We establish that k and w *are* properly rationalizable in the finite version of Crawford's (1982) and Ellingsen and Miettinen's (2008) bilateral commitment bargaining game, by showing that both k and w can be used with positive probability in a proper equilibrium; thus, they are properly rationalizable (Asheim, 2001, Proposition 2). To prove this claim, consider the likelihood orderings $L_1 = \{\{k\}, \{k - 1\}, \ldots, \{\beta_1 + 1\}, \{w\}, \{\beta_1\}, \{\beta_1 - 1\}, \ldots, \{1\}, \{0\}\}$ and $L_2 = \{\{w\}, \{1\}, \{2\}, \ldots, \{\beta_2 - 1\}, \{k\}, \{k - 1\}, \ldots, \{\beta_2 + 1\}, \{\beta_2\}\}$. Since each element in either of these partitions contains only one strategy, they determine a pair of LPSs. It is straightforward to check that this pair of LPSs is a proper equilibrium, according to Blume et al.'s (1991b, Proposition 5) characterization, where player 1 chooses k with probability 1 and player 2 chooses w with probability 1. ■

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	L	R
U	1, 1	1, 1
M	0, 1	2, 0
D	1, 0	0, 1

Dekel-Fudenberg and Proper rationalizability

$R_1^0 = \emptyset$ $\quad\quad$ $R_2^0 = \emptyset$

$R_1^1 = mc\{(D, \{U\})\}$ $\quad\quad$ $R_2^1 = \emptyset$

$\dots$ $\quad\quad$ $\dots$

$R_1^\infty = mc\{(D, \{U\})\}$ $\quad\quad$ $R_2^\infty = \emptyset$

Iterated admissibility

$R_1^0 = \emptyset$ $\quad\quad$ $R_2^0 = \emptyset$

$R_1^1 = mc\{(D, \{U\})\}$ $\quad\quad$ $R_2^1 = \emptyset$

$R_1^2 = mc\{(D, \{U\})\}$ $\quad\quad$ $R_2^2 = mc\{(R, \{L\})\}$

$R_1^3 = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\}$ $\quad\quad$ $R_2^3 = mc\{(R, \{L\})\}$

$\dots$ $\quad\quad$ $\dots$

$R_1^\infty = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\}$ $\quad\quad$ $R_2^\infty = mc\{(R, \{L\})\}$

Figure 1: Iterated admissibility rules out properly rationalizable strategies (G_1).

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
D	1, 0	0, 1

Dekel-Fudenberg

$R_1^0 = \emptyset$ $\qquad$ $R_2^0 = \emptyset$

$R_1^1 = mc\{(D, \{U\})\}$ $\qquad$ $R_2^1 = \emptyset$

$\ldots$ $\qquad$ $\ldots$

$R_1^\infty = mc\{(D, \{U\})\}$ $\qquad$ $R_2^\infty = \emptyset$

Iterated admissibility and Proper rationalizability

$R_1^0 = \emptyset$ $\qquad$ $R_2^0 = \emptyset$

$R_1^1 = mc\{(D, \{U\})\}$ $\qquad$ $R_2^1 = \emptyset$

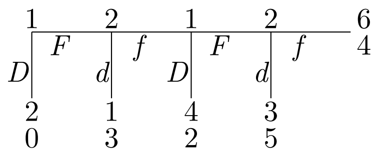
$R_1^2 = mc\{(D, \{U\})\}$ $\qquad$ $R_2^2 = mc\{(R, \{L\})\}$

$R_1^3 = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\}$ $\qquad$ $R_2^3 = mc\{(R, \{L\})\}$

$\ldots$ $\qquad$ $\ldots$

$R_1^\infty = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\}$ $\qquad$ $R_2^\infty = mc\{(R, \{L\})\}$

Figure 2: Iterated admissibility coincides with proper rationalizability (G_2).



	d	fd	ff
D	2, 0	2, 0	2, 0
FD	1, 3	4, 2	4, 2
FF	1, 3	3, 5	6, 4

Dekel-Fudenberg

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$

$$R_1^1 = \emptyset \qquad R_2^1 = mc\{(ff, \{fd\})\}$$

$$R_1^2 = mc\{(FF, \{D, FD\})\} \qquad R_2^2 = mc\{(ff, \{fd\})\}$$

$$\dots \qquad \dots$$

$$R_1^\infty = mc\{(FF, \{D, FD\})\} \qquad R_2^\infty = mc\{(ff, \{fd\})\}$$

Iterated admissibility and Proper rationalizability

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$

$$R_1^1 = \emptyset \qquad R_2^1 = mc\{(ff, \{fd\})\}$$

$$R_1^2 = mc\{(FF, \{FD\})\} \qquad R_2^2 = mc\{(ff, \{fd\})\}$$

$$R_1^3 = mc\{(FF, \{FD\})\} \qquad R_2^3 = mc\{(fd, \{d\}), (ff, \{d\}), (ff, \{fd\})\}$$

$$R_1^4 = mc\{(FD,\{D\}),(FF,\{D\}),(FF,\{FD\})\} \qquad R_2^4 = mc\{(fd, \{d\}), (ff, \{d\}), (ff, \{fd\})\}$$

$$\dots \qquad \dots$$

$$R_1^\infty = mc\{(FD,\{D\}),(FF,\{D\}),(FF,\{FD\})\} \qquad R_2^\infty = mc\{(fd, \{d\}), (ff, \{d\}), (ff, \{fd\})\}$$

Figure 3: A four-legged centipede game (G_3).