Lecture 6:

Common strong belief in rationality

Epistemic foundations for backward induction

8 Common Strong Belief in Rationality

The concept of **common initial belief in rationality** imposes no restrictions on how players revise their beliefs during the game.

What are reasonable ways to **revise** your belief if you observe an **unexpected choice** by an opponent ?

8.1 Example: Father and son

Story: Suppose, you have a twelve year old son.

On the table there are two enormous piles of papers that have to be put in alphabetical order, and this seems a perfect job for him.

You first ask him whether he wants to do the job.

If he accepts, then he will receive a reward if he does the job properly.

However, you will not be at home when your son is doing the job, and checking both piles when you arrive at home requires quite some work.

This may give your son an incentive to cheat, by doing only one pile instead of two.

Utilities for the son:

If you find out that your son has only done one pile, he must do the other pile as well, but will receive no reward.

If you do not catch him cheating, he will get a reward of 5 euros.

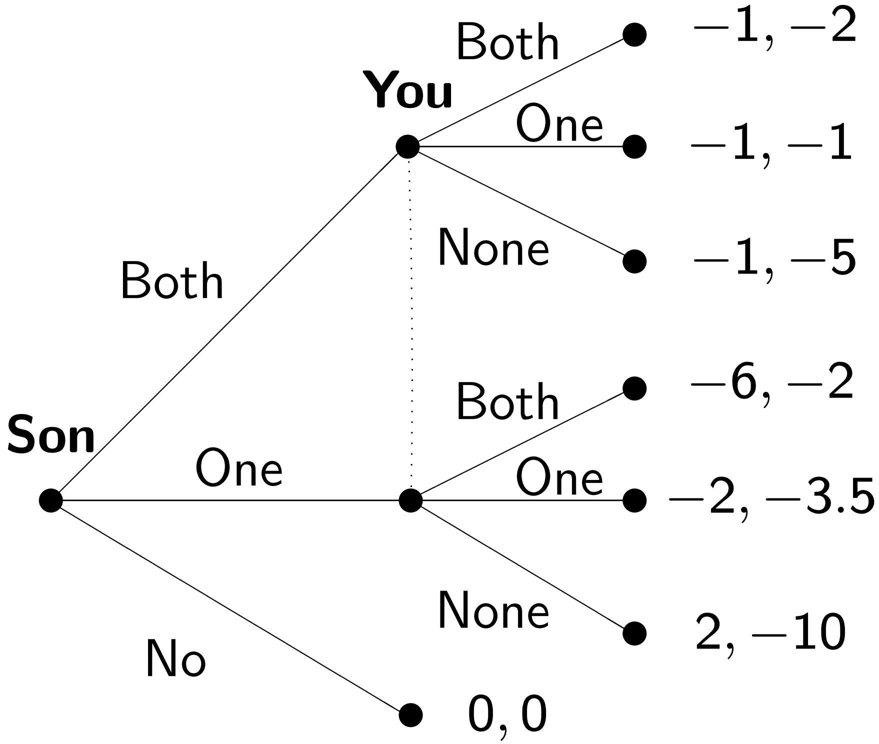
Doing a pile reduces his utility by 3.

Utilities for you:

Checking a pile reduces your utility by 1.

If you reward your son unjustly, this would reduce your utility by 5.

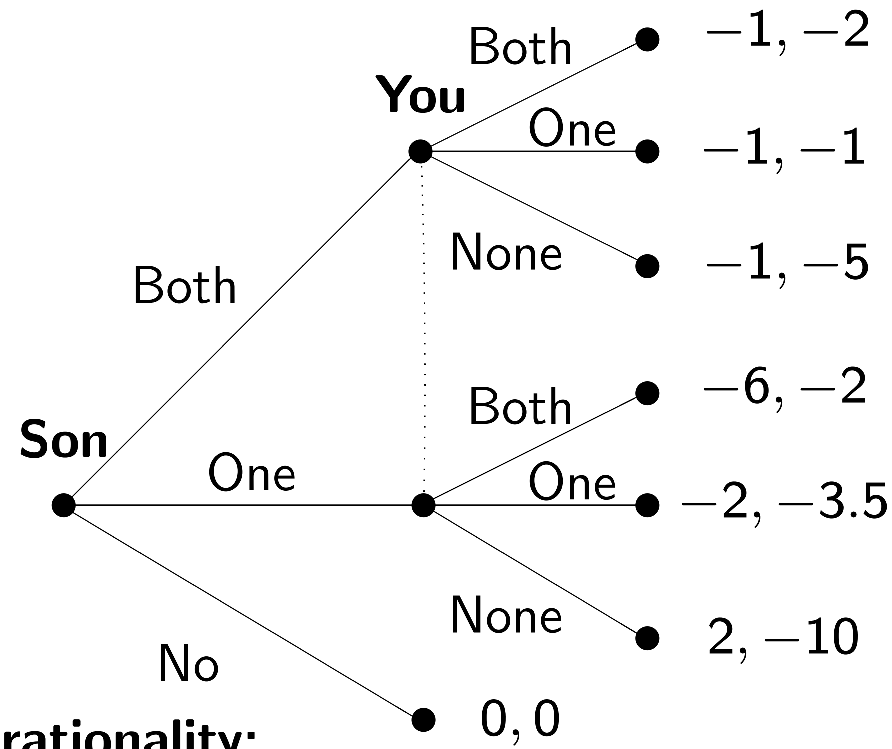
If you check no pile, this would appear immoral to you, and would reduce your utility by 5.



Initially, you believe that your son believes that you will check at least one pile.

Therefore, you initially believe that your son will reject the job.

So, at your information set you must **revise** your belief about your son.



Strong belief in son's rationality:

You conclude that your son must have done **only one pile,** since doing **one pile can be rational** for your son, whereas doing **two piles can never be rational** for him (given the depicted utilities).

Therefore, you will **check both piles.**

8.2 Strong belief in rationality

Intuitively, a player **strongly believes in the opponents' rationality** if, whenever it is **possible** for him to believe that opponent j has chosen a rational strategy, he **must** believe that opponent j has chosen a rational strategy.

In order to formalize this notion, we need a **complete** epistemic model:

Every possible conditional belief hierarchy should be present in our epistemic model.

Consider an **epistemic model** $\mathbf{M} = (T_i, b_i)_{i\in I}$ where

- T_i is the set of types for player i, and

- b_i is a function that assigns to every type $t_i \in T_i$ and to every information set $h_i \in H_i^*$ a conditional belief

$$b_i(t_i, h_i) \in \Delta(S_{-i}(h_i) \times T_{-i}).$$

The epistemic model $\mathbf{M} = (T_i, b_i)_{i\in I}$ is called **complete** if for every possible conditional belief vector $\hat{b}_i = (\hat{b}_i(h_i))_{h_i\in H_i^*}$ you can make, there is a type $t_i \in T_i$ such that $b_i(t_i, h_i) = \hat{b}_i(h_i)$ for all $h_i \in H_i^*$.

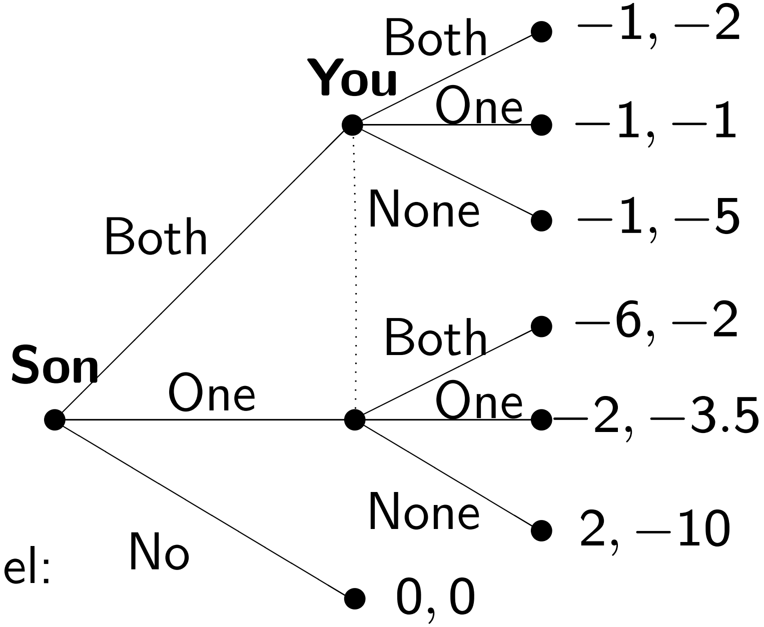
Consider a **complete** epistemic model $\mathbf{M} = (T_i, b_i)_{i \in I}$.

Type t_i **strongly believes in opponent j's rationality** (Battigalli and Siniscalchi, 2002) if at every information set $h_i \in H_i^*$:

if there is some $(\hat{s}_j, \hat{t}_j) \in S_j \times T_j$ where $\hat{s}_j$ leads to h_i and $\hat{s}_j$ is **rational** for $\hat{t}_j$,

then $b_i(t_i, h_i)$ **should** only assign positive probability to strategy-type pairs $(s_j, t_j) \in S_j \times T_j$ where s_j leads to h_i and s_j is **rational** for t_j.

For this definition, it is crucial that the epistemic model is **complete !**

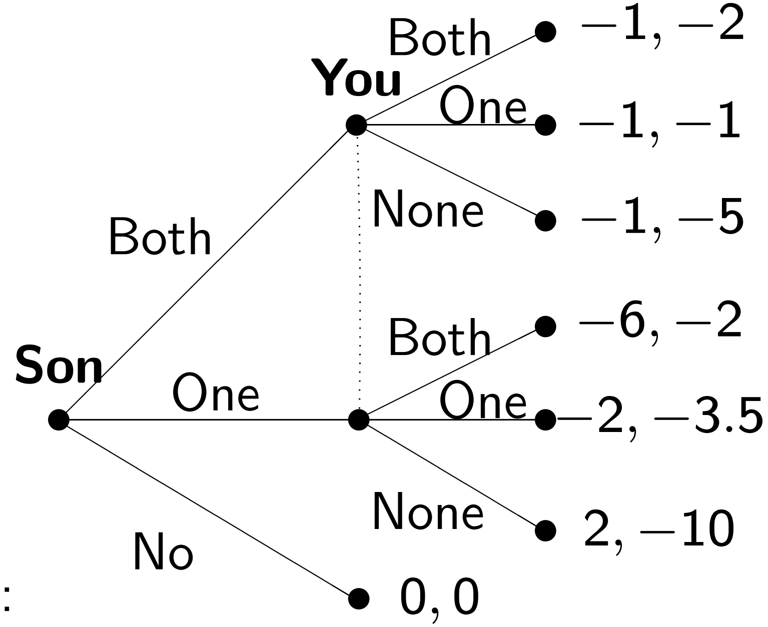


Consider **incomplete** model:

$T_1 = \{t_1^{one}\}$ for you and $T_2 = \{t_2^{no}\}$ for son, where $b_2(t_2^{no}, h_0) = (one, t_1^{one})$.

At your information set h_1, there is no $(\hat{s}_2, \hat{t}_2) \in S_2 \times T_2$ where $\hat{s}_2$ leads to h_1 and $\hat{s}_2$ is rational for $\hat{t}_2$.

So, within this **incomplete** model, strong belief in son's rationality would **not impose restrictions** on your belief at h_1.



Consider **complete** model:

At your information set h_1, there is a $(\hat{s}_2, \hat{t}_2) \in S_2 \times T_2$ where $\hat{s}_2$ leads to h_1 and $\hat{s}_2$ is rational for $\hat{t}_2$, namely $(one, \hat{t}_2)$, where $b_2(\hat{t}_2, h_0) = (none, t_1)$.

So, **strong belief in son's rationality** implies that at h_1, you must believe that son has chosen one.

8.3 Common strong belief in rationality

Intuitive idea:

Consider player i who at information set h_i must form a belief about player j.

If at h_i it is **possible** to believe that player j has chosen a **rational** strategy, and that player j **strongly believes in his opponents' rationality,**

then player i **must** believe at h_i that player j has chosen a **rational** strategy, and that player j **strongly believes in his opponents' rationality.**

Consider a **complete** epistemic model, and a subset of opponent j's types $E_j \subseteq T_j$.

Informally, t_i strongly believes in E_j if, whenever it is **possible** to believe that j chooses rationally while having a type in E_j, type t_i **should** believe that j chooses rationally while having a type in E_j.

Formally, type t_i **strongly believes in** E_j if at every information set $h_i \in H_i^*$:

if there is some $\hat{s}_j \in S_j$ and $\hat{t}_j \in E_j$ such that $\hat{s}_j$ leads to h_i and $\hat{s}_j$ is rational for $\hat{t}_j$,

then type t_i **should** at h_i only assign positive probability to strategy-type pairs (s_j, t_j) in $S_j \times T_j$ where s_j leads to h_i, s_j is rational for t_j and $t_j \in E_j$.

We recursively define the following sets of types:

$$
\begin{aligned}
SBR_i^1 &:= \{t_i \in T_i \mid t_i \text{ strongly believes in the opponents' rationality}\}, \\
SBR_i^2 &:= \{t_i \in SBR_i^1 \mid t_i \text{ strongly believes in } SBR_j^1 \text{ for all } j\}, \\
SBR_i^3 &:= \{t_i \in SBR_i^2 \mid t_i \text{ strongly believes in } SBR_j^2 \text{ for all } j\}, \\
&\vdots \\
SBR_i^k &:= \{t_i \in SBR_i^{k-1} \mid t_i \text{ strongly believes in } SBR_j^{k-1} \text{ for all } j\}, \\
&\vdots
\end{aligned}
$$

Say that type t_i expresses **common strong belief in rationality** (Battigalli and Siniscalchi (2002)) if $t_i \in SBR_i^k$ for all k.

8.4 Example: Watching TV together

Story: You and your partner must decide which TV program to watch this evening: Football or House.

In order to decide, you and your partner must write a program on a piece of paper.

If you both choose the same program, you will watch it together.

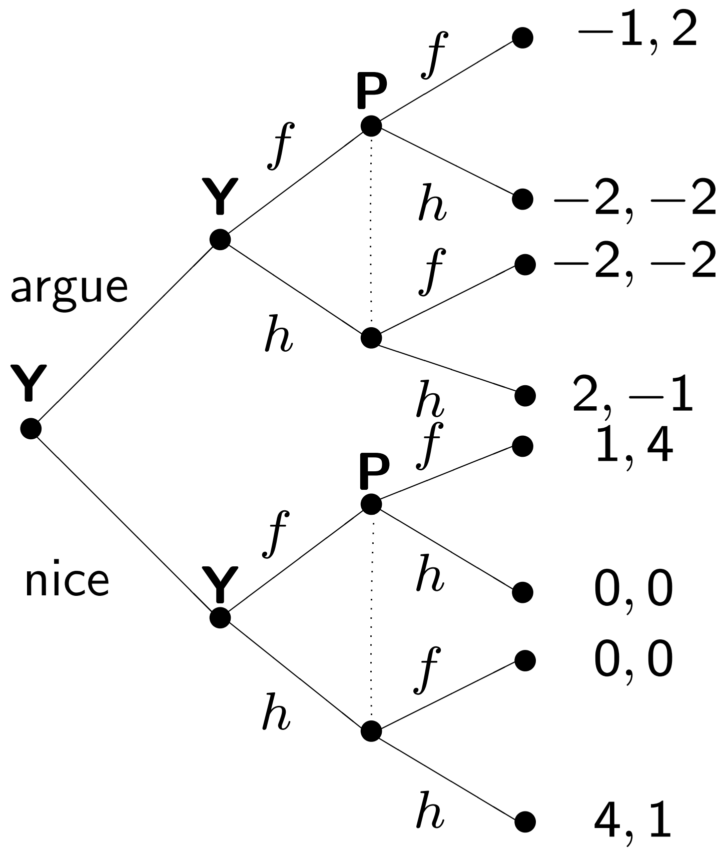
If you write down different programs, you will play a game of Rummycub.

However, before writing something down, you could start to argue with your partner about the program.

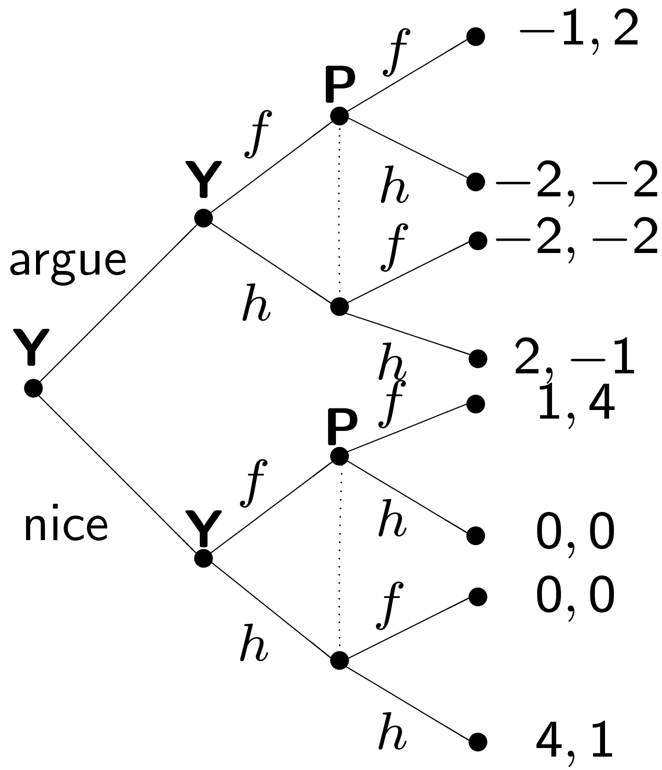
The utilities for you and your partner are as follows:

	Football	House	Rummycub
You	1	4	0
Partner	4	1	0

If you would start to argue, this would reduce your utility and your partner's utility by 2.

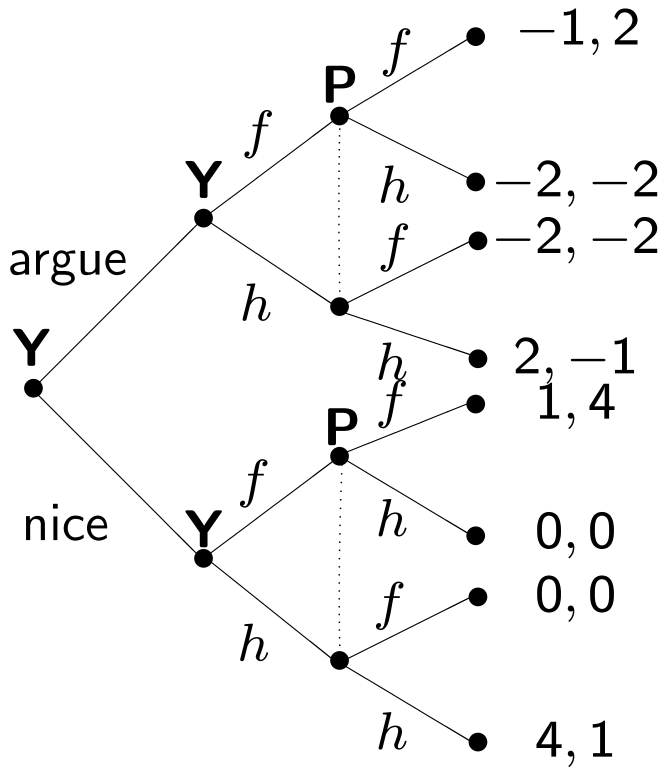


If your partner **strongly believes in your rationality**, then after "argue" your partner should believe that you have chosen "House".

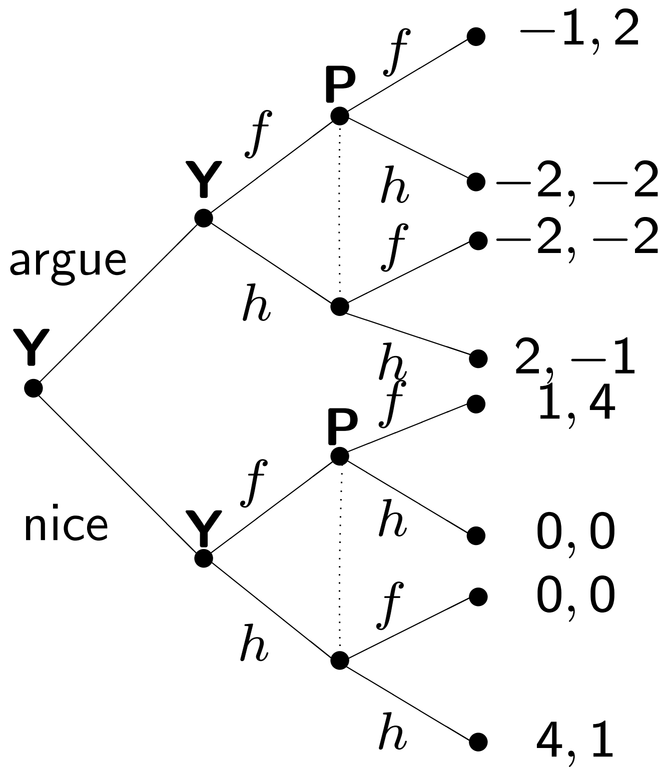


If you **strongly believe in your partner's rationality,** and **strongly believe that your partner strongly believes in your rationality,** you believe that your partner will choose "House" after "argue".

So, you believe to obtain 2 after "argue".

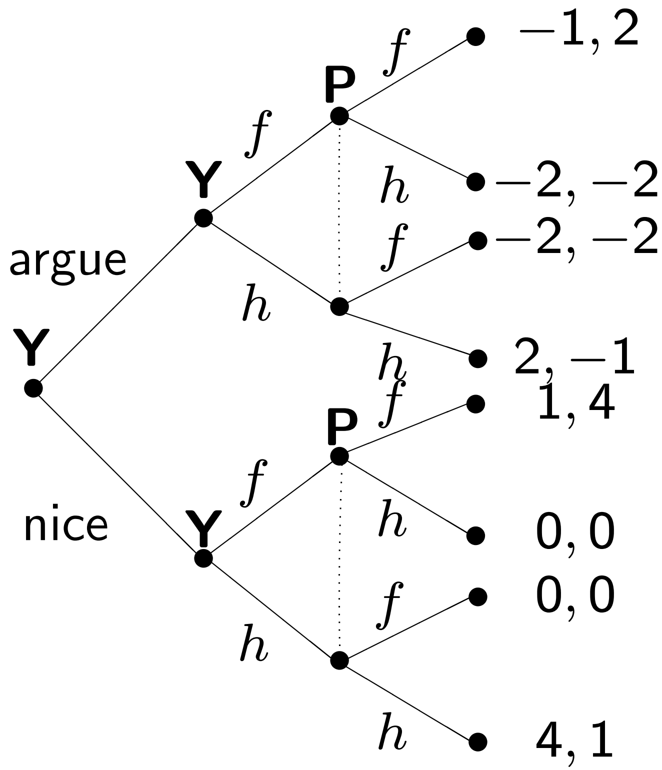


If your partner **strongly believes that you expect to obtain 2 after "argue",** then after "nice" your partner should believe that you have chosen "House".



So, if you **strongly believe in your partner's rationality,** and **strongly believe that your partner strongly believes that you expect to obtain 2 after "argue",** then you believe that your partner will choose "House" after "nice".

So, you should choose "nice" and "House", and expect to obtain 4!



Hence, under **common strong belief in rationality,** you should choose (nice, House), and you would expect to obtain your highest possible utility, namely 4!

8.5 Algorithm

Pearce (1984) has proposed the concept of **extensive form rationalizability.**

His concept is based on an algorithm, that recursively eliminates conditional beliefs and strategies.

The algorithm has later been **simplified** in **Battigalli (1997).**

In **Battigalli and Siniscalchi (2002),** it is shown that this algorithm selects exactly those strategies that can rationally be chosen under **common strong belief in rationality.**

Remember, a **conditional belief** for player i about the opponents' strategies is a vector $b_i = (b_i(h_i))_{h_i \in H_i^*}$ where

$$b_i(h_i) \in \Delta(S_{-i}(h_i))$$

for all information sets $h_i \in H_i^*$.

$B_i^1 :=$ set of all conditional beliefs for player i about the opponents' strategies.

$S_i^1 := \{s_i \in S_i \mid$ there is some $b_i \in B_i^1$ for which s_i is rational$\}$

B_i^2 := set of all conditional beliefs $b_i \in B_i^1$ such that at every information set $h_i \in H_i^*$:

if there is some $s_j \in S_j^1$ that leads to h_i,

then $b_i(h_i)$ **should** only assign positive probability to strategies in S_j^1 that lead to h_i.

$S_i^2 := \{s_i \in S_i^1 \mid$ there is some $b_i \in B_i^2$ for which s_i is rational$\}$.

$B_i^3 :=$ set of all conditional beliefs $b_i \in B_i^2$ such that at every information set $h_i \in H_i^*$:

if there is some $s_j \in S_j^2$ that leads to h_i,

then $b_i(h_i)$ **should** only assign positive probability to strategies in S_j^2 that lead to h_i.

$S_i^3 := \{s_i \in S_i^2 \mid$ there is some $b_i \in B_i^3$ for which s_i is rational$\}$.

Algorithm: Pearce-Battigalli procedure (Pearce (1984), Battigalli (1997))

$B_i^1 :=$ set of all conditional beliefs about the opponents' strategies.

$S_i^1 := \{s_i \in S_i \mid$ there is some $b_i \in B_i^1$ for which s_i is rational$\}$.

$\vdots$

$B_i^k := \{b_i \in B_i^{k-1} \mid$ at every h_i: **if there is** some $s_j \in S_j^{k-1}$ that leads to h_i, then $b_i(h_i)$ **should** only assign positive probability to strategies in S_j^{k-1} that lead to $h_i\}$.

$S_i^k := \{s_i \in S_i^{k-1} \mid$ there is some $b_i \in B_i^k$ for which s_i is rational$\}$.

$\vdots$

A strategy s_i that survives the Pearce-Battigalli procedure is called **extensive form rationalizable.**

Theorem 8.1: (Battigalli and Siniscalchi, 2002)

A strategy s_i can rationally be chosen under **common strong belief in rationality**

if and only if

s_i survives the **Pearce-Battigalli procedure.**

8.6 Proper belief revision

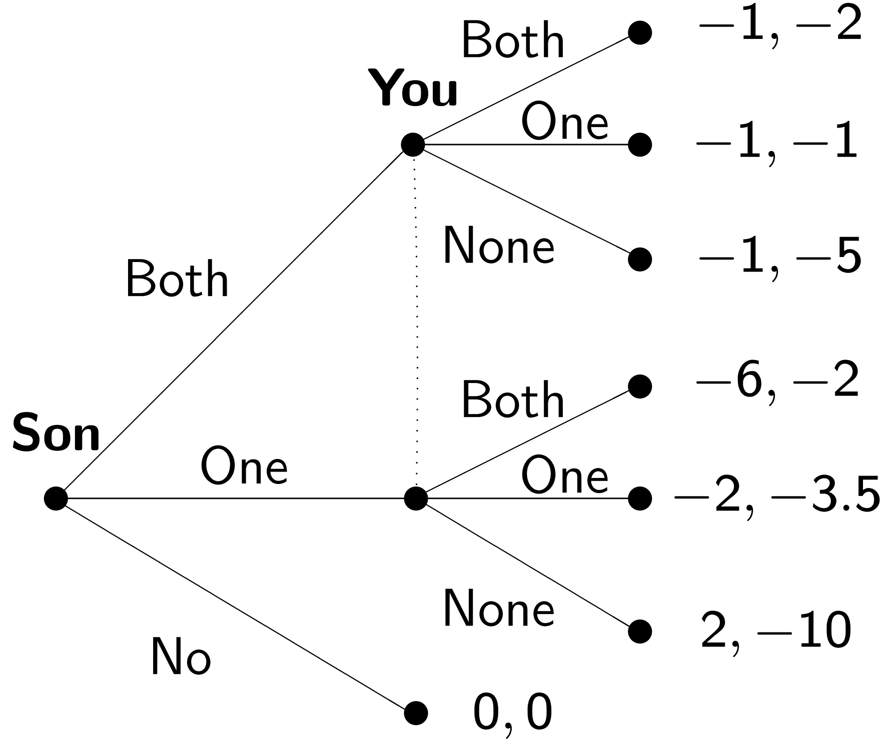
Informal definition of proper belief revision (Perea (2006, 2007)):

Suppose you **initially** believe that the opponent prefers strategy s_j over strategy $\hat{s}_j$.

If the game reaches your information set h_i,

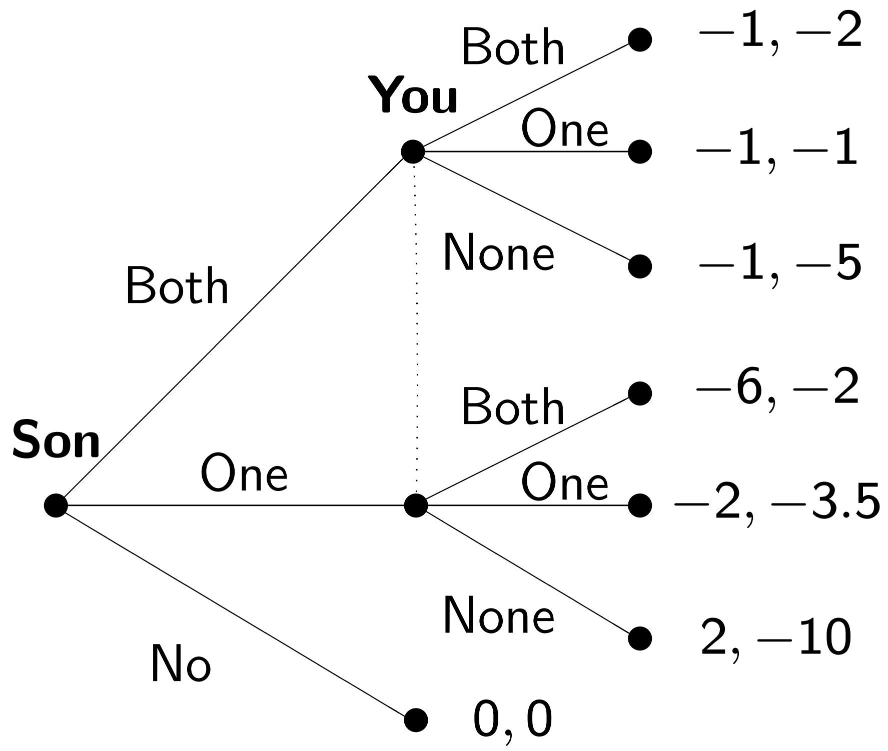
and **both** s_j **and** $\hat{s}_j$ **could have led to** h_i,

then you **still believe** at h_i that the opponent prefers s_j over $\hat{s}_j$.

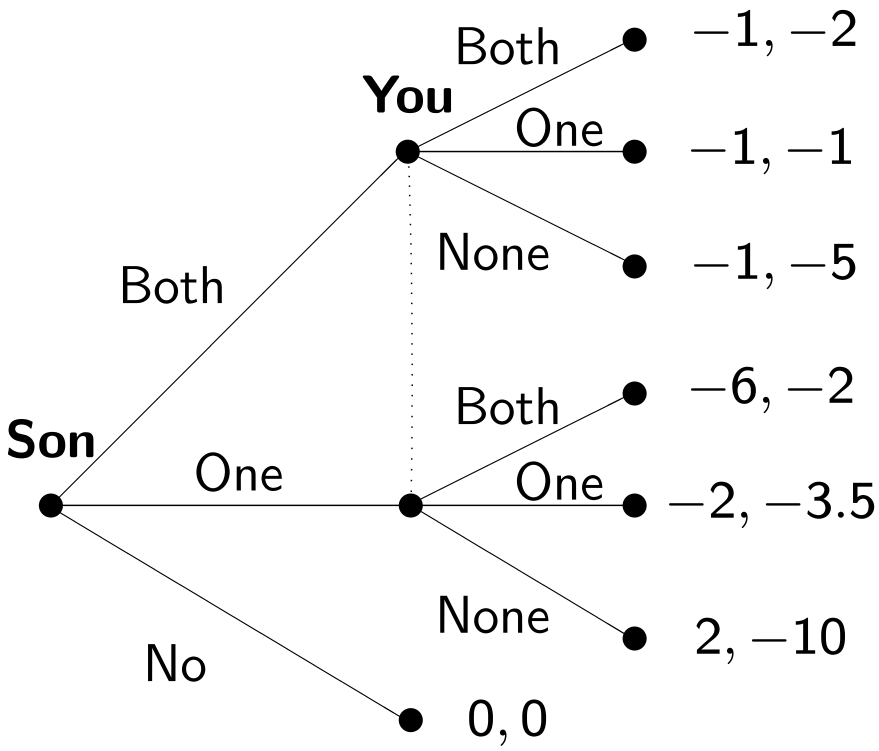


Initially, you believe that your son believes that you will check at least one pile.

So, **initially** you believe that your son prefers “doing both piles” over “doing one pile”.



Since "doing both piles" and "doing one pile" are both possible at your information set, **proper belief revision** requires that at your information set, you **still believe** that your son prefers "doing both piles" over "doing one pile".



So, under **proper belief revision,** you must check only one pile.

We have seen that under **common strong belief in rationality,** you must check both piles.

8.7 Related Models

Shimoji and Watson (1998) propose an algorithm that is equivalent to the Pearce-Battigalli procedure.

Their procedure is based on **conditional dominance** of strategies.

Ben-Porath and Dekel (1992) and **van Damme (1989)** investigate **Burning-Money games**, which are similar to "Watching TV together".

They apply **iterated weak dominance** to Burning-Money games.

Shimoji (2002) shows that in Burning-Money games, **iterated weak dominance** gives the same strategy choices as **common strong belief in rationality.**

8.8 References

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Makoto Shimoji and Joel Watson (1998): "Conditional dominance, rationalizability, and game forms", Journal of Economic Theory 83, 161-195.

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Eric van Damme (1989): "Stable equilibria and forward induction", Journal of Economic Theory 48, 476-496.

9 Epistemic Foundations for Backward Induction

Backward induction is perhaps the oldest idea in game theory.

Zermelo already used the backward induction algorithm in **1913** to prove his famous theorem about chess.

Although the backward induction algorithm may appear completely natural, we need to impose **serious restrictions** on a player's **belief revision policy** in order to arrive at backward induction.

Question: Which restrictions on a player's belief revision policy lead to **backward induction?**

9.1 Example: Kramer versus Kramer

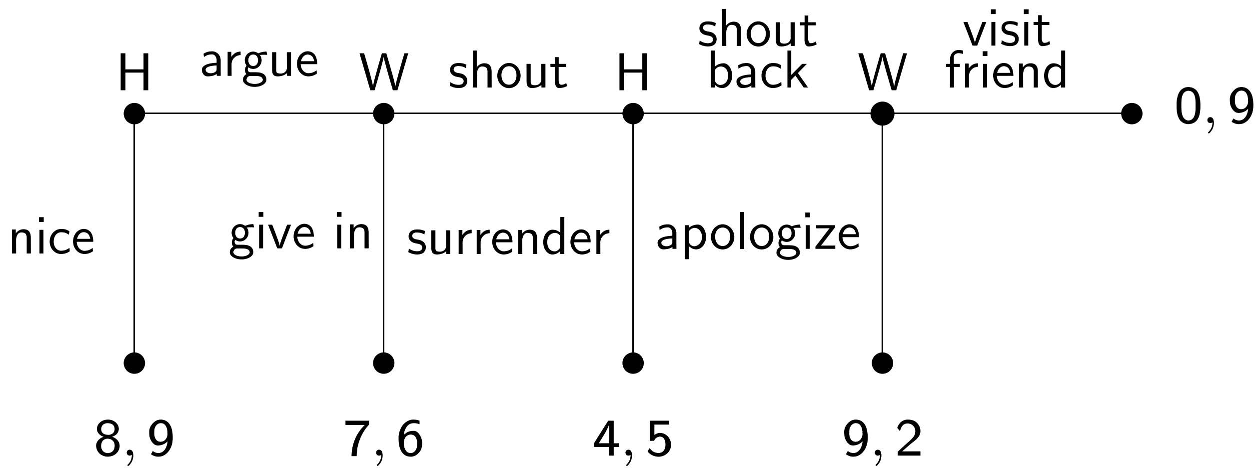
Story: Mr. and Mrs. Kramer must decide which TV program to watch this evening: Football or House.

Their utilities for watching these programs are as follows:

	Football	House
Mr. Kramer	9	8
Mrs. Kramer	8	9

As usual, they enter into a fight before deciding.

Every time the conflict escalates, both their utilities decrease by 2.



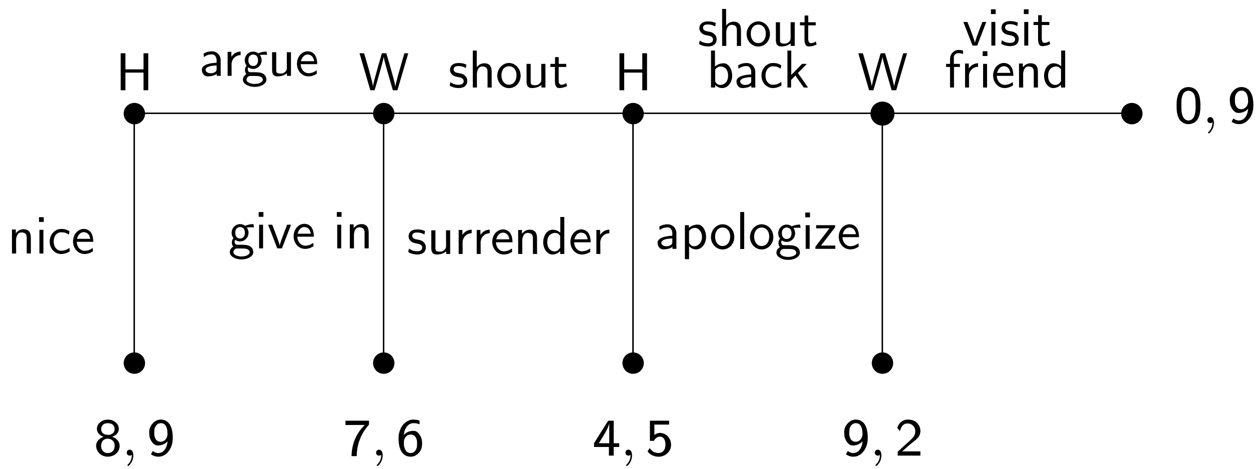
Backward induction:

If the husband shouts back, wife would visit friend.

If the wife shouts, husband would surrender.

If the husband argues, wife would give in.

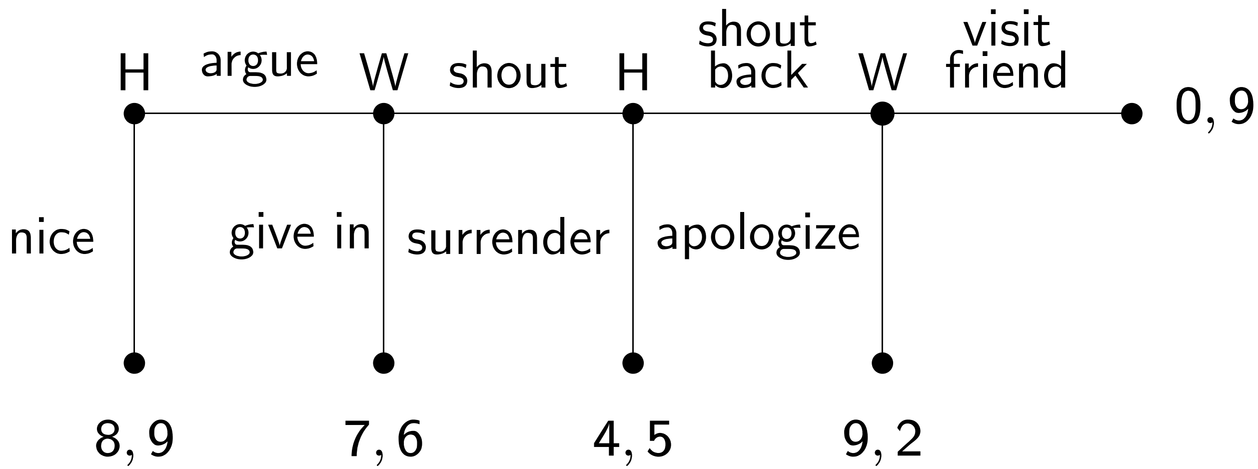
Backward induction strategies: "nice" for husband, and "give in" for wife.



Suppose now that the wife **observes** that the husband has started **arguing.**

Backward induction states that in this case, the wife should **believe** that the husband would **surrender** in the future.

Why should the wife believe this?



Common strong belief in rationality gives a different answer than backward induction:

If the wife observes that the husband has started arguing, she should believe that the husband would **shout back** in the future, since this is the only way for him to get more than 8.

Therefore, the wife should choose "shout, visit friend", and not "give in".

9.2 Overview of epistemic foundations for backward induction

In the literature, many papers have provided many different **epistemic foundations for backward induction.**

Each of these papers imposes restrictions on how a player revises his beliefs.

Every paper contains two main theorems:

Theorem A: If a player revises his beliefs according to the model, then he will choose his backward induction strategy.

Theorem B: The restrictions imposed on the player's belief revision policy do not lead to contradictions.

The different epistemic foundations are often **difficult to compare,** because:

- some use **syntactic models,** others use **semantic models,**

- some semantic models are **state-based (Kripke structures),** others are **type-based,**

- some use **belief operators,** some use **knowledge operators,** and others use yet different operators,

- some use **static** belief (knowledge) operators, others use **dynamic** belief (knowledge) operators,

- some use **conditional beliefs,** others use **lexicographic beliefs,**

- ...

Perea (2007) provides an overview of (most of) these epistemic foundations.

He develops an **epistemic base-model** for dynamic games with perfect information, and **translates** all epistemic foundations **in terms of this base model.**

By doing so, we can **explicitly compare** the restrictions that the various models impose on the player's belief revision policy.

Perea (2007) concentrates on the following papers:

Asheim (2002), Asheim and Perea (2005), Aumann (1995), Balkenborg and Winter (1997), Clausing (2003), Feinberg (2005), Perea (2008), Quesada (2002, 2003), Samet (1996) and Stalnaker (1998).

	Ash	A&P	Aum	Fei1	Per	Que
Common structural belief in event that types ...						
... initially believe in rationality at all inf. sets			*			
... always believe in rationality at future inf. sets that are believed to be reached						*
... always believe in rat. at all future inf. sets	*					
... always believe in rationality at all future and parallel inf. sets		*		*		
... always believe in rationality at all inf. sets					*	

	B&W	Cla	Fei2	Sam	Sta
Common initial belief in event that types ...					
... initially believe in rationality at all inf. sets					*
Forward belief in substantive rationality	*	*	*		
Forward belief in material rationality				*	

	Aum	B&W	Per	Sta
Common structural belief in event that types ...				
... do not revise belief about opponents' choices and beliefs at future and parallel inf. sets	*	*		*
... minimally revise belief about opponents' utilities and beliefs			*	

Among these conditions,

- the **weakest** set of conditions is **forward belief in material rationality** (Samet (1996)), and

- the **strongest** set of conditions is **common structural belief in the event that types always believe in rationality at all future and parallel information sets** (Asheim and Perea (2005) and Feinberg 1 (2005)).

We show:

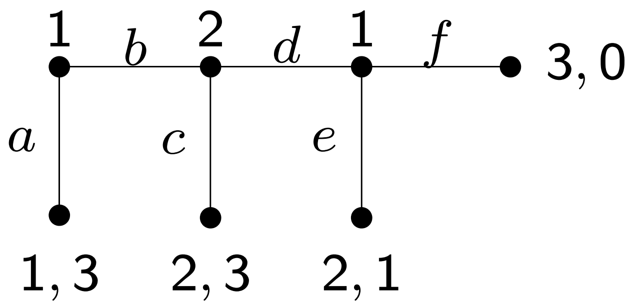
The **weakest** set of conditions leads to backward induction.

The **strongest** set of conditions is always possible.

9.3 Backward induction

Consider a finite dynamic game Γ with **perfect information.**

That is, a player, whenever he has to choose, knows exactly which choices have been made by his opponents until then.



The game Γ is **free of ties** if for every player i, every information set $h_i \in H_i$, every two choices $c_i, c_i' \in C_i(h_i)$, every terminal node z following c_i, and every terminal node z' following c_i', it holds that $u_i(z) \neq u_i(z')$.

Assume, from now on, that the dynamic game with perfect information is free of ties.

We define, for every information set h_i, the unique **backward induction choice** $c^*(h_i)$ as follows:

- If h_i is an **ultimate information set,** that is, h_i is not followed by any other information set, then $c^*(h_i)$ is the unique choice that maximizes i's utility at h_i.

- If h_i is a **penultimate information set,** that is, h_i is only followed by ultimate information sets, $c^*(h_i)$ is the unique choice at h_i that maximizes i's utility at h_i, given that the backward induction choices are made at all (ultimate) information sets that follow h_i.

- And so on.

Player i's **backward induction strategy** is the strategy s_i^* with $s_i^*(h_i) = c^*(h_i)$ for all $h_i \in H_i(s_i^*)$.

9.4 Forward belief in material rationality

Let Γ be a finite dynamic game with perfect information that is free of ties.

Consider a **finite epistemic model** $(T_i, b_i)_{i \in I}$ for Γ.

Type t_i **believes at** $h_i \in H_i^*$ **that player** j **chooses rationally at** $h_j \in H_j^*$ if $b_i(t_i, h_i)$ only assigns positive probability to strategy-type pairs (s_j, t_j) where s_j is optimal for t_i at h_j.

Type t_i **believes that** h_j **can be reached from** h_i if h_j follows h_i, and $b_i(t_i, h_i)$ assigns positive probability to an opponents' strategy profile s_{-i} that leads to h_j.

Let $H(t_i, h_i)$ be the set of information sets which t_i believes can be reached from h_i.

Type t_i expresses **forward belief in material rationality** if:

- type t_i believes at every h_i that every opponent j chooses rationally at every $h_j \in H(t_i, h_i)$,

- type t_i assigns at every h_i only positive probability to types t_j who at every $h_j \in H(t_i, h_i)$ believe that every opponent k chooses rationally at every $h_k \in H(t_j, h_j)$,

- and so on.

Theorem 9.1: (Samet (1996))

Consider a finite dynamic game with perfect information that is free of ties. Let $(T_i, b_i)_{i \in I}$ be a finite epistemic model.

Let t_i be a type that expresses **forward belief in material rationality.**

Then, there is **only one** strategy that is **rational** for t_i, namely player i's **backward induction strategy.**

Proof:

Let t_i be a type that expresses **forward belief in material rationality.**

We show: At every $h_i \in H_i$, type t_i must make the backward induction choice $c^*(h_i)$.

Case 1: Suppose that h_i is an **ultimate** information set. Then the optimal choice for t_i at h_i is $c^*(h_i)$.

Case 2: Suppose that h_i is a **penultimate information set.**

At h_i, type t_i believes that every ultimate information set h_j following h_i can be reached from h_i.

Since t_i expresses **forward belief in material rationality,** t_i believes at h_i that every opponent j chooses rationally at every ultimate information set h_j that follows h_i.

So, t_i believes at h_i that every opponent j, at every ultimate information set h_j following h_i, makes the backward induction choice $c^*(h_j)$.

Therefore, the optimal choice for t_i at h_i is $c^*(h_i)$.

Case 3: Suppose that h_i is **only followed by penultimate and ultimate information sets.**

Let $H^1(h_i)$ be the set of information sets that **immediately follow** h_i, and let $H^2(h_i)$ be those information sets that **follow** h_i **within two steps.**

Type t_i believes that every $h_j \in H^1(h_i)$ can be reached from h_i.

Take $h_j \in H^1(h_i)$. Since t_i expresses **forward belief in material rationality,** t_i believes at h_i that:

- j chooses rationally at h_j, and that

- j believes at h_j that every opponent chooses rationally at every ultimate information set following h_j.

So, t_i believes at h_i that j chooses $c^*(h_j)$ at h_j.

Hence, t_i believes at h_i that at every $h_j \in H^1(h_i)$ the backward induction choice will be made.

Now, take $h_j \in H^2(h_i)$. Then, h_j must be an ultimate information set.

It is possible that t_i believes that h_j cannot be reached from h_i.

However, if t_i believes that h_j **can be reached** from h_i, then t_i must believe that j chooses rationally at h_j.

So, t_i believes at h_i that at every $h_j \in H^2(h_i)$, believed to be reached from h_i, the backward induction choice will be made.

Summarizing: Type t_i believes at h_i that at every h_j, believed to be reached from h_i, the backward induction choice will be made.

But then, the optimal choice for t_i at h_i is the backward induction choice.

And so on.

This completes the proof.

9.5 Belief in rationality at future and parallel information sets

Type t_i **always believes in rationality at future and parallel information sets** if at every $h_i \in H_i^*$, type t_i believes that every opponent j chooses rationally at every $h_j \in H_j^*$ that does not precede h_i.

Type t_i expresses **common structural belief in the event that types always believe in rationality at future and parallel information sets,** if:

- t_i always believes in rationality at future and parallel information sets,

- t_i always believes that his opponents believe so,

- and so on.

Theorem 9.2: (Asheim and Perea (2005), Feinberg (2005))

Consider a finite dynamic game with perfect information that is free of ties. Then, we can construct an epistemic model $(T_i, b_i)_{i \in I}$ such that every type expresses **common structural belief in the event that types always believe in rationality at future and parallel information sets.**

Proof:

For every opponent j, and information set $h_i \in H_i^*$, let $s_j^*(h_i)$ be the strategy that:

- at every h_j preceding h_i, prescribes the unique choice that leads to h_i, and

- at every h_j that does not precede h_i, prescribes the backward induction choice $c^*(h_j)$.

Define sets of types $T_i := \{t_i\}$ for every player i, where

$$b_i(t_i, h_i) = ((s_j^*(h_i), t_j))_{j \neq i}$$

for every $h_i \in H_i^*$.

Then, for every player i, type t_i **always believes in rationality at future and parallel information sets.**

Therefore, for every player i, type t_i expresses **common structural belief in the event that types always believe in rationality at future and parallel information sets.**

This completes the proof.

9.6 References

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