Lecture 4:

Proper rationalizability

An epistemic foundation for iterated weak dominance

5 Proper Rationalizability

In a setting with **lexicographic beliefs,** it is possible to deem the opponent's choice a **infinitely more likely than** the opponent's choice b, without completely discarding b.

In the previous lecture, we discussed **common weak belief in cautiousness and rationality.**

This concept only imposes restrictions on your **first-order belief,** but **imposes no restrictions at all on your higher-order beliefs!**

Idea in proper rationalizability:

Restrictions should also be imposed on higher-order beliefs:

If you believe that your opponent prefers his choice a over his choice b, then you should deem a **infinitely more likely than** b.

Hence, if you believe that your opponent prefers choice a over b, and prefers b over c, then you should deem a **infinitely more likely than** b, and b **infinitely more likely than** c.

5.1 Example: To which pub shall I go?

Story: You and your friend Barbara will both go out this evening. There are three pubs in the village: a, b and c.

The preferences for you and Barbara are as follows:

	Pub a	Pub b	Pub c
You	1	1	1
Barbara	3	2	1

Barbara likes talking to you, but you have a headache.

If you both go the same pub: -1 for you, and $+1$ for Barbara.

Barbara

		Pub a	Pub b	Pub c
You	Pub a	$0,4$	$1,2$	$1,1$
	Pub b	$1,3$	$0,3$	$1,1$
	Pub c	$1,3$	$1,2$	$0,2$

If you are **cautious,** and express **common weak belief in cautiousness and rationality,** you can choose either Pub b or Pub c:

Take $b_1(t_1^b) = \begin{bmatrix} (a, t_2^a) \\ (c, t_2^a) \\ (b, t_2^a) \end{bmatrix}$, $b_1(t_1^c) = \begin{bmatrix} (a, t_2^a) \\ (b, t_2^a) \\ (c, t_2^a) \end{bmatrix}$ and $b_2(t_2^a) = \begin{bmatrix} (b, t_1^b) \\ (a, t_1^b) \\ (c, t_1^b) \end{bmatrix}$.

All types are cautious, and express common weak belief in cautiousness and rationality.

Barbara

		Pub a	Pub b	Pub c
You	Pub a	$0,4$	$1,2$	$1,1$
	Pub b	$1,3$	$0,3$	$1,1$
	Pub c	$1,3$	$1,2$	$0,2$

However, if you believe that Barbara is **cautious,** you should believe that Barbara prefers Pub a over Pub b, and Pub b over Pub c.

Hence, if you are **cautious,** and **respect Barbara's preferences,** then you should deem Pub a **infinitely more likely** than Pub b, and Pub b **infinitely more likely** than Pub c.

Therefore, you should go to Pub c, and not to Pub b!

5.2 Respecting the opponent's preferences

Let us first review some definitions from the previous lecture.

Consider a finite game $\Gamma = (C_i, u_i)_{i \in I}$ with two players.

A finite **epistemic model with lexicographic beliefs** is a tuple $\mathbf{M} = (T_i, b_i)_{i \in I}$ where

- T_i is the finite set of types for player i, and
- b_i is a function that assigns to every type $t_i \in T_i$ a **lexicographic belief** $b_i(t_i)$ on $C_j \times T_j$.

Let $T_j(t_i) \subseteq T_j$ be the set of opponent's types t_j to which $b_i(t_i)$ assigns positive probability at some of its levels.

So, $t_j \in T_j(t_i)$ if there is some $k \in \{1, ..., K\}$ and $c_j \in C_j$ with $b_i^k(c_j, t_j) > 0$.

Type t_i is **cautious** if for every opponent's type $t_j \in T_j(t_i)$ and **every opponent's choice** $c_j \in C_j$ there is some belief b_i^k in $b_i(t_i)$ with $b_i^k(c_j, t_j) > 0$.

Consider a cautious type t_i.

Intuitively**,** t_i **respects the opponent's preferences** if, whenever t_i believes that j prefers choice c_j over choice $\hat{c}_j$, then t_i deems c_j **infinitely more likely than** $\hat{c}_j$.

Formally**,** t_i **respects the opponent's preferences** (Asheim, 2001) if, for every $t_j \in T_j(t_i)$, and for every $c_j, \hat{c}_j \in C_j$ where

t_j **prefers** c_j over $\hat{c}_j$

it holds that

t_i deems (c_j, t_j) **infinitely more likely than** $(\hat{c}_j, t_j)$.

Remember: Type t_i **weakly believes in j's rationality** if the **first-order belief** b_i^1 only assigns positive probability to pairs (c_j, t_j) where c_j is rational for t_j.

Observation:

If t_i **respects the opponent's preferences,** then t_i **weakly believes in j's rationality.**

Namely, consider a type $t_j \in T_j(t_i)$ and a choice c_j that is **not rational** for t_j.

Then, there is some $\hat{c}_j$ such that t_j prefers $\hat{c}_j$ over c_j. Hence, t_i should deem (c_j, t_j) infinitely less likely than $(\hat{c}_j, t_j)$.

So, the first-order belief b_i^1 cannot assign positive probability to (c_j, t_j).

5.3 Proper rationalizability

Let t_i be a type with lexicographic belief

$$b_i(t_i) = (b_i^1, ..., b_i^K)$$

on $C_j \times T_j$.

Let $E_j \subseteq T_j$ be a subset of opponent's types. Remember:

Type t_i **fully believes** in E_j if **every** b_i^k in $b_i(t_i)$ only assigns positive probability to types t_j in E_j.

Take subset of types $E \subseteq T_i \cup T_j$.

Type t_i expresses **common full belief** in E if:

- t_i fully believes in E,
- t_i fully believes that j fully believes in E,
- t_i fully believes that j fully believes that i fully believes in E,

and so on.

A type t_i with lexicographic beliefs is called **properly rationalizable** (Schuhmacher (1999), Asheim (2001)) if:

- t_i is cautious,
- t_i respects the opponent's preferences,
- t_i expresses **common full belief** in cautiousness,
- t_i expresses **common full belief** in the event that types respect the opponent's preferences.

A **choice** c_i is called **properly rationalizable** if there is an epistemic model $\mathbf{M} = (T_i, b_i)_{i \in I}$ with lexicographic beliefs, and a **properly rationalizable type** $t_i \in T_i$ such that c_i is rational for t_i.

Observation:

Every type t_i that is properly rationalizable expresses, in particular, **common weak belief in cautiousness and rationality.**

Hence, every properly rationalizable choice survives the Dekel-Fudenberg procedure.

5.4 Example: To which pub shall I go?

Barbara

		Pub a	Pub b	Pub c
You	Pub a	$0, 4$	$1, 2$	$1, 1$
	Pub b	$1, 3$	$0, 3$	$1, 1$
	Pub c	$1, 3$	$1, 2$	$0, 2$

Consider a type t_1 for you.

If t_1 **fully believes that opponent is cautious,** then t_1 must believe that every $t_2 \in T_2(t_1)$ prefers a over b, and prefers b over c.

So, for every $t_2 \in T_2(t_1)$, type t_1 must deem (a, t_2) **infinitely more likely than** (b, t_2), and must deem (b, t_2) **infinitely more likely than** (c, t_2).

Barbara

You		Pub a	Pub b	Pub c
	Pub a	$0, 4$	$1, 2$	$1, 1$
	Pub b	$1, 3$	$0, 3$	$1, 1$
	Pub c	$1, 3$	$1, 2$	$0, 2$

So, every properly rationalizable type t_1 must choose Pub c.

Hence, there is only one candidate for a properly rationalizable choice for you, namely Pub c.

Barbara

		Pub a	Pub b	Pub c
You	Pub a	$0,4$	$1,2$	$1,1$
	Pub b	$1,3$	$0,3$	$1,1$
	Pub c	$1,3$	$1,2$	$0,2$

Consider epistemic model $T_1 = \{t_1^c), T_2 = \{t_2^a\}$ with

$$b_1(t_1^c) = \begin{bmatrix} (a, t_2^a) \\ (b, t_2^a) \\ (c, t_2^a) \end{bmatrix} \text{ and } b_2(t_2^a) = \begin{bmatrix} (c, t_1^c) \\ (b, t_1^c) \\ (a, t_1^c) \end{bmatrix}.$$

Then, t_1^c and t_2^a are **cautious** and **respect the opponent's preferences.**

So, t_1^c is a **properly rationalizable type,** and Pub c is a **properly rationalizable choice** for you.

Now, suppose that Barbara values your presence at 2, instead of 1.

	Pub a	Pub b	Pub c
Pub a	$0,5$	$1,2$	$1,1$
Pub b	$1,3$	$0,4$	$1,1$
Pub c	$1,3$	$1,2$	$0,3$

Consider epistemic model $T_1 = \{t_1^b, t_1^c\}$, $T_2 = \{t_2^a, t_2^b\}$ where

$$b_1(t_1^b) = \begin{bmatrix} (a, t_2^a) \\ (c, t_2^a) \\ (b, t_2^a) \end{bmatrix}, \; b_1(t_1^c) = \begin{bmatrix} (b, t_2^b) \\ (a, t_2^b) \\ (c, t_2^b) \end{bmatrix},$$

$$b_2(t_2^a) = \begin{bmatrix} (c, t_1^c) \\ (a, t_1^c) \\ (b, t_1^c) \end{bmatrix}, \; b_2(t_2^b) = \begin{bmatrix} (b, t_1^b) \\ (c, t_1^b) \\ (a, t_1^b) \end{bmatrix}.$$

	Pub a	Pub b	Pub c
Pub a	$0, 5$	$1, 2$	$1, 1$
Pub b	$1, 3$	$0, 4$	$1, 1$
Pub c	$1, 3$	$1, 2$	$0, 3$

$$t_1^b \sim \begin{bmatrix} (a, t_2^a) \\ (c, t_2^a) \\ (b, t_2^a) \end{bmatrix}, t_1^c \sim \begin{bmatrix} (b, t_2^b) \\ (a, t_2^b) \\ (c, t_2^b) \end{bmatrix}, t_2^a \sim \begin{bmatrix} (c, t_1^c) \\ (a, t_1^c) \\ (b, t_1^c) \end{bmatrix}, t_2^b \sim \begin{bmatrix} (b, t_1^b) \\ (c, t_1^b) \\ (a, t_1^b) \end{bmatrix}.$$

Then, all types are cautious and respect the opponent's preferences.

Hence, t_1^b and t_1^c are **properly rationalizable types.**

Therefore, both Pub b and Pub c are **properly rationalizable choices** for you.

	Pub a	Pub b	Pub c
Pub a	$0, 5$	$1, 2$	$1, 1$
Pub b	$1, 3$	$0, 4$	$1, 1$
Pub c	$1, 3$	$1, 2$	$0, 3$

Pub a **cannot** be a properly rationalizable choice for you:

If Barbara is cautious, she must prefer Pub a over Pub c (but not necessarily Pub a over Pub b!).

So, if you **respect the opponent's preferences,** then you must deem Pub a **infinitely more likely** than Pub c.

Hence, Pub c is always better for you than Pub a.

5.5 Existence

Theorem 5.1. (Schuhmacher (1999), Asheim (2001))

Let $\Gamma = (C_i, u_i)_{i \in I}$ be a finite, static game with two players.

Then, there is an epistemic model $\mathbf{M} = (T_i, b_i)_{i \in I}$ with lexicographic beliefs, such that T_i contains a **properly rationalizable type.**

Illustration of proof.

Consider the following game:

	d	e	f
a	$0,5$	$1,2$	$1,1$
b	$1,3$	$0,4$	$1,1$
c	$1,3$	$1,2$	$0,3$

Lexicographic belief $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ induces preference relation $\begin{bmatrix} c \\ b \\ a \end{bmatrix}$ over choices.

We write: $\begin{bmatrix} d \\ e \\ f \end{bmatrix} \to \begin{bmatrix} c \\ b \\ a \end{bmatrix}$.

	d	e	f
a	0, 5	1, 2	1, 1
b	1, 3	0, 4	1, 1
c	1, 3	1, 2	0, 3

$$\begin{bmatrix} d \\ e \\ f \end{bmatrix} \to \begin{bmatrix} c \\ b \\ a \end{bmatrix} \to \underbrace{\begin{bmatrix} d \\ f \\ e \end{bmatrix} \to \begin{bmatrix} b \\ c \\ a \end{bmatrix} \to \begin{bmatrix} e \\ d \\ f \end{bmatrix} \to \begin{bmatrix} c \\ a \\ b \end{bmatrix} \to \begin{bmatrix} d \\ f \\ e \end{bmatrix}}.$$

Make $T_1 = \{t_1^{bca}, t_1^{cab}\}$ and $T_2 = \{t_2^{dfe}, t_2^{edf}\}$ where $t_1^{bca} \sim \begin{bmatrix} (d, t_2^{dfe}) \\ (f, t_2^{dfe}) \\ (e, t_2^{dfe}) \end{bmatrix}$,

$$t_1^{cab} \sim \begin{bmatrix} (e, t_2^{edf}) \\ (d, t_2^{edf}) \\ (f, t_2^{edf}) \end{bmatrix}, \; t_2^{dfe} \sim \begin{bmatrix} (c, t_1^{cab}) \\ (a, t_1^{cab}) \\ (b, t_1^{cab}) \end{bmatrix}, \; t_2^{edf} \sim \begin{bmatrix} (b, t_1^{bca}) \\ (c, t_1^{bca}) \\ (a, t_1^{bca}) \end{bmatrix}.$$

	d	e	f
a	0, 5	1, 2	1, 1
b	1, 3	0, 4	1, 1
c	1, 3	1, 2	0, 3

$$t_1^{bca} \sim \begin{bmatrix} (d, t_2^{dfe}) \\ (f, t_2^{dfe}) \\ (e, t_2^{dfe}) \end{bmatrix}, \; t_1^{cab} \sim \begin{bmatrix} (e, t_2^{edf}) \\ (d, t_2^{edf}) \\ (f, t_2^{edf}) \end{bmatrix},$$

$$t_2^{dfe} \sim \begin{bmatrix} (c, t_1^{cab}) \\ (a, t_1^{cab}) \\ (b, t_1^{cab}) \end{bmatrix}, \; t_2^{edf} \sim \begin{bmatrix} (b, t_1^{bca}) \\ (c, t_1^{bca}) \\ (a, t_1^{bca}) \end{bmatrix}.$$

All types in T_1 and T_2 are properly rationalizable.

5.6 Related Models

The idea of **respecting the opponent's preferences** is already present in **Myerson (1978).**

He defines the concept of **proper equilibrium,** by taking sequences $(\mu_i^n)_{n \in \mathbf{N}}$ of probability distributions with full support on C_j.

The main condition is: If c_j is a better choice under μ_j^n than $\hat{c}_j$, then

$$\lim_{n \to \infty} \frac{\mu_i^n(\hat{c}_j)}{\mu_i^n(c_j)} = 0.$$

Blume, Brandenburger and Dekel (1991) have characterized **proper equilibrium** by means of lexicographic beliefs.

Perea and Asheim (2008) provide an **algorithm** that computes all properly rationalizable choices in a two-player game.

5.7 References

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Lawrence Blume, Adam Brandenburger and Eddie Dekel (1991): "Lexicographic probabilities and equilibrium refinements", Econometrica 59, 81-98.

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6 An Epistemic Foundation for Iterated Weak Dominance

The algorithm of **iterated weak dominance** is a very old idea in game theory.

Already in the fifties people used the algorithm as a way to **eliminate "unreasonable choices"** that could not be ruled out by Nash equilibrium.

However, until 2008 there was **no real justification** for using this algorithm.

Brandenburger, Friedenberg and Keisler (2008) finally provided an **epistemic foundation** for iterated weak dominance.

6.1 Iterated Weak Dominance

Consider a finite static game $\Gamma = (C_i, u_i)_{i \in I}$ with two players, i and j.

Take a subset $D_j \subseteq C_j$ of choices for player j.

Say that choice c_i is **weakly dominated** for player i on D_j if there is some randomized choice $\mu_i \in \Delta(C_i)$ with

$u_i(c_i, c_j) \leq u_i(\mu_i, c_j)$ for every $c_j \in D_j$, and

$u_i(c_i, c_j) < u_i(\mu_i, c_j)$ for at least some $c_j \in D_j$.

Algorithm: Iterated weak dominance.

$$
\begin{aligned}
C_i^1 &:= \{c_i \in C_i \mid c_i \text{ not } \textbf{weakly} \text{ dominated on } C_j\} \\
C_i^2 &:= \{c_i \in C_i^1 \mid c_i \text{ not } \textbf{weakly} \text{ dominated on } C_j^1\} \\
C_i^3 &:= \{c_i \in C_i^2 \mid c_i \text{ not } \textbf{weakly} \text{ dominated on } C_j^2\} \\
&\vdots \\
C_i^k &:= \{c_i \in C_i^{k-1} \mid c_i \text{ not } \textbf{weakly} \text{ dominated on } C_j^{k-1}\} \\
&\vdots
\end{aligned}
$$

Why is there a connection between iterated weak dominance and lexicographic beliefs?

- We have seen: A choice is **not weakly dominated** if and only if it is optimal for some **cautious lexicographic belief.**

- If, in the process of iterated weak dominance, you **eliminate** a weakly dominated choice c_j, it is as if you deem c_j **infinitely less likely** than any surviving choice at that stage.

6.2 Example: To which pub shall I go?

Barbara

You		Pub a	Pub b	Pub c
	Pub a	$0,4$	$1,2$	$1,1$
	Pub b	$1,3$	$0,3$	$1,1$
	Pub c	$1,3$	$1,2$	$0,2$

Iterated weak dominance:

Pub b and Pub c are weakly dominated for Barbara by Pub a.

So, eliminate Pub b and Pub c for Barbara.

Barbara

You		Pub a
	Pub a	$0, 4$
	Pub b	$1, 3$
	Pub c	$1, 3$

In reduced game, Pub a is weakly dominated for you by Pub b and Pub c.

So, eliminate Pub a for you.

	Pub a
Pub b	$1, 3$
Pub c	$1, 3$

Hence, your choices Pub b and Pub c survive iterated weak dominance.

Remember that **proper rationalizability** uniquely selected your choice Pub c.

What causes this difference?

Barbara

		Pub a	Pub b	Pub c
You	Pub a	$0, 4$	$1, 2$	$1, 1$
	Pub b	$1, 3$	$0, 3$	$1, 1$
	Pub c	$1, 3$	$1, 2$	$0, 2$

Proper rationalizability requires that you deem Pub a **infinitely more likely** than Pub b, and that you deem Pub b **infinitely more likely** than Pub c.Therefore you should **not** choose Pub b.

Iterated weak dominance, by eliminating Pub b and Pub c for Barbara at the first step, requires that you deem Pub b and Pub c **infinitely less likely than** Pub a. However, it does **not** require than you deem Pub b **infinitely more likely** than Pub c. Therefore, you **can** choose Pub b.

Now, suppose that Barbara values your presence at 2, instead of 1.

	Pub a	Pub b	Pub c
Pub a	$0, 5$	$1, 2$	$1, 1$
Pub b	$1, 3$	$0, 4$	$1, 1$
Pub c	$1, 3$	$1, 2$	$0, 3$

Iterated weak dominance:

For Barbara, Pub c is weakly dominated by Pub a.

Eliminate Pub c for Barbara.

	Pub a	Pub b
Pub a	$0, 5$	$1, 2$
Pub b	$1, 3$	$0, 4$
Pub c	$1, 3$	$1, 2$

In reduced game, Pub a and Pub b are weakly dominated for you by Pub c.

Eliminate Pub a and Pub b for you.

	Pub a	Pub b
Pub c	$1, 3$	$1, 2$

So, **iterated weak dominance** selects a unique choice for you, namely Pub c.

We have seen that **proper rationalizability** allows you to choose Pub b **and** Pub c.

What causes this difference?

	Pub a	Pub b	Pub c
Pub a	$0, 5$	$1, 2$	$1, 1$
Pub b	$1, 3$	$0, 4$	$1, 1$
Pub c	$1, 3$	$1, 2$	$0, 3$

In **iterated weak dominance,** by eliminating Barbara's choice Pub c, you should deem Pub c **infinitely less likely** than Pub a and Pub b. Therefore, you should **not** choose Pub b.

In **proper rationalizability,** you should deem Pub c **infinitely less likely** than Pub a, but **not necessarily** infinitely less likely than Pub b. Therefore, you **can** choose Pub b.

6.3 Assuming versus believing

	Pub a	Pub b	Pub c
Pub a	$0,5$	$1,2$	$1,1$
Pub b	$1,3$	$0,4$	$1,1$
Pub c	$1,3$	$1,2$	$0,3$

In **iterated weak dominance,** after eliminating Barbara's choice Pub c, we can eliminate your choices Pub a and Pub b because these are **weakly dominated on** $D_2 = \{\text{Pub } a, \text{ Pub } b\}$.

Lemma 6.1:

Consider a finite static game $\Gamma = (C_i, u_i)_{i \in I}$ with two players, i and j. Let $D_j \subseteq C_j$ be a subset of j's choices. Then, a choice c_i is **not weakly dominated** on C_j and **not weakly dominated** on D_j

if and only

c_i is optimal for some cautious lexicographic belief b_i on C_j that deems every choice in D_j **infinitely more likely** than every choice in $C_j \backslash D_j$.

Let $b_i = (b_i^1, ..., b_i^K)$ be a cautious lexicographic belief on C_j. Let $D_j \subseteq C_j$ be a subset of j's choices.

We say that b_i **assumes** D_j if b_i deems every choice in D_j **infinitely more likely** than **every** choice in $C_j \backslash D_j$.

Example: Let $C_2 = \{a, b, c\}$, and let $b_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Then, b_1 assumes $\{a\}$, but b_1 does **not** assume $\{a, c\}$.

So, "assume" is a **non-monotonic** belief operator!

Note that b_1 **weakly believes** in $\{a\}$ and in $\{a, c\}$!

6.4 Assuming the opponent's rationality

First two steps in iterated weak dominance (from player i's viewpoint):

1. Select set C_j^1 of j's choices that are **not weakly dominated on** C_i.

2. Select those choices for player i that are **not weakly dominated on** C_j **and not weakly dominated on** C_j^1.

By Lemma 6.1, step 2 requires that i **assumes** that j chooses from C_j^1.

Set C_j^1 contains exactly those choices that are rational for some cautious type.

Hence, step 2 requires that i **assumes that** j **chooses rationally.**

Consider a cautious type t_i with lexicographic belief $b_i(t_i) = (b_i^1, ..., b_i^K)$ on $C_j \times T_j$.

Say that type t_i **assumes j's rationality** if $b_i(t_i)$ deems every **rational** choice-type pair (c_j, t_j) in $C_j \times T_j$ **infinitely more likely** than every **irrational** choice-type pair $(\hat{c}_j, \hat{t}_j)$ in $C_j \times T_j$.

For this definition to "work", it is necessary that the set T_j of player j types contains **all belief hierarchies one can possibly construct!**

That is, the epistemic model must be **complete!**

	d	e	f
a	$2,2$	$1,1$	$0,5$
b	$1,2$	$2,1$	$0,5$
c	$0,0$	$0,6$	$0,5$

In **iterated weak dominance,** you eliminate the weakly dominated choice d at the first step, but not e and f.

Hence, in iterated weak dominance, you must deem e **infinitely more likely** than d.

Therefore, under iterated weak dominance, you must choose b.

	d	e	f
a	$2,2$	$1,1$	$0,5$
b	$1,2$	$2,1$	$0,5$
c	$0,0$	$0,6$	$0,5$

Consider the **incomplete** epistemic model $\mathbf{M}$ with $T_1 = \{t_1\}$, $T_2 = \{t_2\}$ where

$$b_1(t_1) = \begin{bmatrix} (f, t_2) \\ (d, t_2) \\ (e, t_2) \end{bmatrix} \text{ and } b_2(t_2) = \begin{bmatrix} (a, t_1) \\ (b, t_1) \\ (c, t_1) \end{bmatrix}.$$

Then, within $\mathbf{M}$, types t_1 and t_2 **assume the opponent's rationality.**

But t_1 does **not** deem e infinitely more likely than d, as iterated weak dominance requires!

Consider an **epistemic model with lexicographic beliefs** $\mathrm{M} = (T_i, b_i)_{i \in I}$ where

- T_i is the set of types for player i, and
- b_i is a function that assigns to every type $t_i \in T_i$ a **lexicographic belief** $b_i(t_i)$ on $C_j \times T_j$.

We say that the epistemic model M is **complete** if for **every** lexicographic belief $\hat{b}_i$ on $C_j \times T_j$ there is a type $t_i \in T_i$ with $b_i(t_i) = \hat{b}_i$, and similarly for player j.

Hence, **every possible belief hierarchy** must be present in the model M!

Note that a **complete** epistemic model $\mathbf{M}$ must contain **infinitely many types** for both players, since there are infinitely many possible belief hierarchies.

Theorem 6.2: (Brandenburger, Friedenberg and Keisler (2008))

For every finite, static two-player game there is a **complete** epistemic model with lexicographic beliefs.

Consider a **complete** epistemic model $\mathbf{M} = (T_i, b_i)_{i \in I}$ with lexicographic beliefs.

Consider a type $t_i \in T_i$ with lexicographic belief $b_i(t_i) = (b_i^1, ..., b_i^K)$ on $C_j \times T_j$.

Say that a type $t_i \in T_i$ has **full support** if for every $(c_j, t_j) \in C_j \times T_j$ there is some $k \in \{1, ..., K\}$ such that $b_i^k(c_j, t_j) > 0$.

In particular, every type with full support is **cautious.**

We say: Type t_i **assumes that** j **has full support** if t_i deems every type t_j with full support infinitely more likely than every type $\hat{t}_j$ without full support.

Lemma 6.3:

Let C_j^1 be the set of j's choices that are not weakly dominated on C_i.

Suppose that $\mathbf{M} = (T_i, b_i)_{i \in I}$ is a **complete** epistemic model with lexicographic beliefs.

Take a type $t_i \in T_i$ with **full support,** which **assumes** that [j has **full support** and **chooses rationally**].

Then t_i deems every choice in C_j^1 infinitely more likely than every choice in $C_j \backslash C_j^1$.

Proof of Lemma 6.3:

Take a choice $c_j \in C_j^1$ and a choice $\hat{c}_j \in C_j \backslash C_j^1$.

Since the epistemic model $\mathbf{M}$ is complete, c_j is rational for some type $t_j^{c_j} \in T_j$ with full support.

On the other hand, $\hat{c}_j$ is **not** rational for any type $t_j \in T_j$ with full support.

Since t_i has full support, and assumes that [j has full support and chooses rationally] t_i must deem $(c_j, t_j^{c_j})$ infinitely more likely than any pair $(\hat{c}_j, t_j)$ which contains $\hat{c}_j$.

Hence, t_i must deem c_j infinitely more likely than $\hat{c}_j$.

6.5 k-th order assumption of rationality

Let $\mathbf{M} = (T_i, b_i)_{i \in I}$ be a complete epistemic model, and $t_i \in T_i$ a type with full support.

Suppose that t_i has lexicographic belief $b_i(t_i) = (b_i^1, ..., b_i^K)$ on $C_j \times T_j$.

Take a subset of types $E_j \subseteq T_j$.

Say that type t_i **assumes the event** E_j if t_i deems every type $t_j \in E_j$ infinitely more likely than every type $\hat{t}_j \in T_j \backslash E_j$.

We recursively define the following sets of types:

$$
\begin{aligned}
AR_i^1 &:= \{t_i \in T_i \mid t_i \text{ has full support and assumes } j\text{'s rationality}\}, \\
AR_i^2 &:= \{t_i \in AR_i^1 \mid t_i \text{ assumes } AR_j^1\} \\
AR_i^3 &:= \{t_i \in AR_i^2 \mid t_i \text{ assumes } AR_j^2\}, \\
&\vdots \\
AR_i^k &:= \{t_i \in AR_i^{k-1} \mid t_i \text{ assumes } AR_j^{k-1}\}
\end{aligned}
$$

We say that t_i expresses k-**th order assumption of rationality** if $t_i \in AR_i^k$.

Problem: In general,

$$AR_j^1 \underset{\neq}{\supset} AR_j^2 \underset{\neq}{\supset} AR_j^3 \underset{\neq}{\supset} \ ... \ AR_j^k \underset{\neq}{\supset} AR_j^{k+1} \underset{\neq}{\supset} \ ...$$

So, **common** assumption of rationality would require that you deem:

- AR_j^1 infinitely more likely than $T_j \backslash AR_j^1$,

- AR_j^2 infinitely more likely than $AR_j^1 \backslash AR_j^2$,

- AR_j^3 infinitely more likely than $AR_j^2 \backslash AR_j^3$,

- and so on, ad infinitum.

Since, in general, all the events $AR_j^1, AR_j^2, ...$ are **different,** this is not possible in a lexicographic belief $b_i = (b_i^1, b_i^2, ..., b_i^K)$ on $C_j \times T_j$!

So, in general, **common** assumption of rationality is impossible!

However, k-th order assumption of rationality is always possible for every **finite** k!

Theorem 6.4: (Brandenburger, Friedenberg and Keisler (2008))

Let $\Gamma = (C_i, u_i)_{i \in I}$ be a finite, static game with two players. Let $\mathbf{M} = (T_i, b_i)_{i \in I}$ be a **complete** epistemic model with lexicographic beliefs. Let k be the number of steps needed in iterated weak dominance.

Then, a choice c_i can rationally be chosen by a full support type $t_i \in T_i$ that expresses k**-th order assumption of rationality**

if and only if

c_i survives **iterated weak dominance.**

Idea of the proof:

Take a full support type t_i that expresses k-th order assumption of rationality.

Show: t_i must make a choice c_i that survives iterated weak dominance.

$$C_j^1 := \{c_j \in C_j \mid c_j \text{ not weakly dominated on } C_i\}.$$

Since t_i has full support, and assumes that [j has full support and chooses rationally], we know, by Lemma 6.3, that t_i must assume that j chooses from C_j^1.

$$C_j^2 := \{c_j \in C_j^1 \mid c_j \text{ not weakly dominated on } C_i^1\}.$$

Since t_i assumes the event that [j has full support, chooses rationally and assumes i's rationality], t_i must assume that [j has full support, chooses rationally and assumes that i chooses from C_i^1].

Therefore, t_i must assume that j chooses from C_j^2.

And so on.

So, eventually t_i must assume that j chooses from C_j^k.

Hence, t_i must deem all choices in C_j^k infinitely more likely than all choices in $C_j \backslash C_j^k$.

As such, t_i must choose from C_i^k.

6.6 Reference

Adam Brandenburger, Amanda Friedenberg and H. Jerome Keisler (2008): "Admissibility in games", Econometrica 76, 307-352.