Part II: Lexicographic beliefs in static games

Lecture 3:

Common weak belief in rationality and the Dekel-Fudenberg procedure

4 Common weak belief in rationality and the Dekel-Fudenberg procedure

In this lecture, we introduce the assumption that you **reason cautiously** about your opponents:

Although you may deem some opponents' choices more likely than others, you **do not completely rule out any opponent's choice.**

Motivation: There is always a possibility that the opponent makes a mistake, or that you are wrong about the opponent's utility function.

4.1 Example: Should I call her or not?

Story: Your friend Barbara will go out this evening. There are only two pubs in the village, a and b, and the preferences for you and Barbara are given by the following table:

	Pub a	Pub b
Your utility	1	0
Barbara's utility	0	1

You have to decide whether you want to **call her or not.**

You will only join if she goes to pub a.

If you don't join, you stay at home alone and your utility would be 0.

Barbara

		Pub a	Pub b
You	Call	$1, 0$	$0, 1$
	Don't call	$0, 0$	$0, 1$

If you **reason cautiously**, your **unique best choice** should be to **call.**

Can we model this with **standard beliefs?**

If you **believe in Barbara's rationality**, you should assign **probability 0** to her choosing Pub a.

However, if you **reason cautiously**, you should assign **positive probability** to her choosing Pub a.

We have a contradiction here!

So, within a model of **standard beliefs,** it is not possible to combine

belief in the opponent's rationality

with

reasoning cautiously.

Hence, we have to **extend** our notion of beliefs to a **richer structure.**

4.2 Lexicographic beliefs: The main idea

Barbara

		Pub a	Pub b
You	Call	$1, 0$	$0, 1$
	Don't call	$0, 0$	$0, 1$

Is it possible to deem Pub b **infinitely more likely than** Pub a, without completely discarding the possibility that Barbara chooses Pub a ?

Yes: Use **lexicographic beliefs** rather than standard beliefs.

Barbara

You		Pub a	Pub b
	Call	$1, 0$	$0, 1$
	Don't call	$0, 0$	$0, 1$

Write your belief about Barbara's choice as follows:

$$b_1 = \begin{bmatrix} \text{Pub } b \\ \text{Pub } a \end{bmatrix}.$$

Read: You deem Pub b **infinitely more likely than** Pub a, but you still deem Pub a **possible.**

Suppose, your opponent has set of choices $\{a, b, c, d, e\}$.

Example of lexicographic belief:

$$b_1 = \begin{bmatrix} \frac{1}{2}a + \frac{1}{2}b \\ \frac{1}{3}c + \frac{1}{3}d + \frac{1}{3}e \end{bmatrix}.$$

Interpretation:

You deem a and b equally likely.

You deem a and b infinitely more likely than c, d and e.

You deem c, d and e equally likely.

Example of lexicographic belief:

$$b_1 = \begin{bmatrix} \frac{1}{2}a + \frac{1}{2}b \\ c \\ \frac{1}{3}d + \frac{2}{3}e \end{bmatrix}.$$

Interpretation:

You deem a and b equally likely.

You deem a and b infinitely more likely than c.

You deem c infinitely more likely than d and e.

You deem e twice as likely as d.

Example of lexicographic belief:

$$b_1 = \begin{bmatrix} \frac{1}{2}a + \frac{1}{2}b \\ \frac{1}{4}a + \frac{3}{4}c \\ \frac{1}{4}b + \frac{1}{2}d + \frac{1}{4}e \end{bmatrix}.$$

Interpretation:

You deem a and b equally likely.

You deem a and b infinitely more likely than c.

You deem c infinitely more likely than d and e.

You deem d twice as likely as e.

4.3 Lexicographic beliefs: Definition

Consider a finite static game with two players, i and j. Let C_j be set of choices for player j.

A **lexicographic belief** (Blume, Brandenburger and Dekel, 1991) for player i about j's choice is a **list of probability distributions**

$$b_i = (b_i^1, b_i^2, ..., b_i^K)$$

where $b_i^k \in \Delta(C_j)$ for every $k \in \{1, ..., K\}$.

Consider a lexicographic belief $b_i = (b_i^1, b_i^2, ..., b_i^K)$.

Define: $\text{supp}(b_i^k) = \{c_j \mid b_i^k(c_j) > 0\}$.

Interpretation:

Player i deems all choices in $\text{supp}(b_i^1)$ **infinitely more likely** than all choices in

$$\text{supp}(b_i^2) \setminus \text{supp}(b_i^1).$$

Player i deems all choices in $\text{supp}(b_i^2)$ **infinitely more likely** than all choices in

$$\text{supp}(b_i^3) \setminus (\text{supp}(b_i^2) \cup \text{supp}(b_i^1)).$$

Etcetera.

Every lexicographic belief induces a **list of expected utilities** for each of your choices:

	a	b	c	d	e
f	3	2	1	0	3
g	0	3	2	1	0
h	0	0	3	2	1

If lex. belief is $b_1 = \begin{bmatrix} \frac{1}{2}a + \frac{1}{2}b \\ \frac{1}{3}c + \frac{1}{3}d + \frac{1}{3}e \end{bmatrix}$, then

$u_1(f, b_1) = \begin{bmatrix} 2.5 \\ 1.33 \end{bmatrix}$, $u_1(g, b_1) = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$, and $u_1(h, b_1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Choice f is **optimal** under lex. belief b_1.

	a	b	c	d	e
f	3	2	1	0	3
g	0	3	2	1	0
h	0	0	3	2	1

If lex. belief is $b_1 = \begin{bmatrix} \frac{2}{5}a + \frac{3}{5}c \\ \frac{1}{3}b + \frac{1}{3}d + \frac{1}{3}e \end{bmatrix}$, then

$u_1(f, b_1) = \begin{bmatrix} 1.8 \\ 1.67 \end{bmatrix}$, $u_1(g, b_1) = \begin{bmatrix} 1.2 \\ 1.33 \end{bmatrix}$, and $u_1(h, b_1) = \begin{bmatrix} 1.8 \\ 1 \end{bmatrix}$.

Choice f is **optimal** under lex. belief b_1.

	a	b	c	d	e
f	3	2	1	0	3
g	0	3	2	1	0
h	0	0	3	2	1

If lex. belief is $b_1 = \begin{bmatrix} \frac{2}{5}a + \frac{3}{5}c \\ \frac{1}{2}b + \frac{1}{2}d \\ e \end{bmatrix}$, then

$$u_1(f, b_1) = \begin{bmatrix} 1.8 \\ 1 \\ 3 \end{bmatrix}, \; u_1(g, b_1) = \begin{bmatrix} 1.2 \\ 2 \\ 0 \end{bmatrix}, \text{ and } u_1(h, b_1) = \begin{bmatrix} 1.8 \\ 1 \\ 1 \end{bmatrix}.$$

Choice f is **optimal** under lex. belief b_1.

4.4 Optimal choices

Suppose that player i holds lexicographic belief

$$b_i = (b_i^1, b_i^2, ..., b_i^K)$$

about j's choice.

Every choice $c_i \in C_i$ leads to **list of expected utilities**

$$u_i(c_i, b_i) = \begin{bmatrix} u_i(c_i, b_i^1) \\ u_i(c_i, b_i^2) \\ \vdots \\ u_i(c_i, b_i^K) \end{bmatrix}.$$

Suppose that player i holds lexicographic belief

$$b_i = (b_i^1, b_i^2, ..., b_i^K)$$

about j's choice.

Consider two choices c_i and $\hat{c}_i$. Player i **prefers** c_i over $\hat{c}_i$ if

$$u_i(c_i, b_i^1) > u_i(\hat{c}_i, b_i^1), \qquad \text{or}$$

$$u_i(c_i, b_i^1) = u_i(\hat{c}_i, b_i^1), \quad u_i(c_i, b_i^2) > u_i(\hat{c}_i, b_i^2), \quad \text{or}$$

$$\begin{array}{l} u_i(c_i, b_i^1) = u_i(\hat{c}_i, b_i^1), \\ u_i(c_i, b_i^2) = u_i(\hat{c}_i, b_i^2), \end{array} \quad u_i(c_i, b_i^3) > u_i(\hat{c}_i, b_i^3), \quad \text{or}$$

$$\vdots$$

Suppose that player i holds lexicographic belief

$$b_i = (b_i^1, b_i^2, ..., b_i^K)$$

about j's choice.

Choice c_i is **optimal** for player i under the lexicographic belief b_i if there is no other choice $\hat{c}_i$ which he prefers over c_i.

4.5 Epistemic Model

In an epistemic model with **standard beliefs,** every type t_i holds a **standard probabilistic belief** $b_i(t_i) \in \Delta(C_j \times T_j)$ on the opponent's choice-type pairs.

In an epistemic model with **lexicographic beliefs,** every type t_i holds a **lexicographic belief** $b_i(t_i)$ on the set $C_j \times T_j$ of opponent's choice-type pairs.

A **lexicographic belief** b_i on $C_j \times T_j$ is a list

$$b_i = (b_i^1, b_i^2, ..., b_i^K)$$

of probability distributions, where $b_i^k \in \Delta(C_j \times T_j)$ for every $k \in \{1, ..., K\}$.

The probability distribution b_i^1 is called the **first-order belief**, b_i^2 is the **second-order belief,** and so on.

Consider a finite game $\Gamma = (C_i, u_i)_{i \in I}$ with two players.

A finite **epistemic model with lexicographic beliefs** is a tuple $\mathbf{M} = (T_i, b_i)_{i \in I}$ where

- T_i is the finite set of types for player i, and

- b_i is a function that assigns to every type $t_i \in T_i$ a **lexicographic belief** $b_i(t_i)$ on $C_j \times T_j$.

Let t_i be a type with lexicographic belief

$$b_i(t_i) = (b_i^1, ..., b_i^K)$$

on $C_j \times T_j$.

Choice c_i is **rational** for t_i if c_i is optimal for the lexicographic belief that t_i holds on C_j.

Intuitively, type t_i is **cautious** if the lexicographic belief takes into account each of the opponent's choices.

Formally, let $T_j(t_i) \subseteq T_j$ be the set of opponent's types t_j to which $b_i(t_i)$ assigns positive probability at some of its levels.

So, $t_j \in T_j(t_i)$ if there is some $k \in \{1, ..., K\}$ and $c_j \in C_j$ with $b_i^k(c_j, t_j) > 0$.

Type t_i is **cautious** if for every opponent's type $t_j \in T_j(t_i)$ and **every opponent's choice** $c_j \in C_j$ there is some belief b_i^k in $b_i(t_i)$ with $b_i^k(c_j, t_j) > 0$.

4.6 How to define belief in opponent's rationality?

Recall: In a model with **standard beliefs,** type t_i has a **single** probability distribution over $C_j \times T_j$.

To believe in j's rationality means that $b_i(t_i)$ only assigns positive probability to pairs (c_j, t_j) where c_j is rational for t_j.

Now, consider a type t_i with **lexicographic belief** $b_i(t_i) = (b_i^1, ..., b_i^K)$ on $C_j \times T_j$.

Belief in the opponent's rationality can be defined in at least two different ways:

Type t_i **fully believes in j's rationality** if **every** b_i^k in $b_i(t_i)$ only assigns positive probability to pairs (c_j, t_j) where c_j is rational for t_j.

Type t_i **weakly believes in j's rationality** if the **first-order belief** b_i^1 only assigns positive probability to pairs (c_j, t_j) where c_j is rational for t_j.

Problem: It is **impossible** to simultaneously impose

- that type t_i is **cautious,** and

- that type t_i **fully believes in** j's **rationality.**

Namely, if t_i is cautious, then it must also assign positive probability to **irrational choices** at some of its levels $b_i^1, b_i^2, ..., b_i^K$, so it cannot fully believe in j's rationality.

However, it is **possible** to impose that t_i is cautious, and **weakly believes in** j's **rationality.**

4.7 Common weak belief in rationality

Let t_i be a type with lexicographic belief

$$b_i(t_i) = (b_i^1, ..., b_i^K).$$

Let $E_j \subseteq T_j$ be a subset of the opponent's types.

For instance, E_j is set of j's types that weakly believe in i's rationality, or E_j is set of j's types that are cautious.

Type t_i **weakly believes** in E_j if the **first-order belief** b_i^1 only assigns positive probability to types t_j in E_j.

Type t_i **fully believes** in E_j if **every** b_i^k in $b_i(t_i)$ only assigns positive probability to types t_j in E_j.

Type t_i expresses **common weak belief in rationality** if

- t_i weakly believes in j's rationality,
- t_i weakly believes that j weakly believes in i's rationality,
- t_i weakly believes that j weakly believes that i weakly believes in j's rationality, and so on.

Type t_i expresses **common weak belief in cautiousness** if

- t_i weakly believes that j is cautious,
- t_i weakly believes that j weakly believes that i is cautious,
- t_i weakly believes that j weakly believes that i weakly believes that j is cautious, and so on.

In the remainder of this lecture, we concentrate on types t_i that

- are cautious,
- express common weak belief in cautiousness,
- express common weak belief in rationality.

Is this combination of conditions always possible?

What choices can rationally be made by such types?

Is there an algorithm that computes these choices?

4.8 Example: Teaching a lesson

Story: It is Friday, and teacher will give a surprise exam next week.

In order to pass, you must study last two days before exam.

In order to write a perfect exam, and gain a compliment by your father, you must study for six days.

Utilities for you: Pass: $+5$, Studying one day: -1, Compliment: $+4$

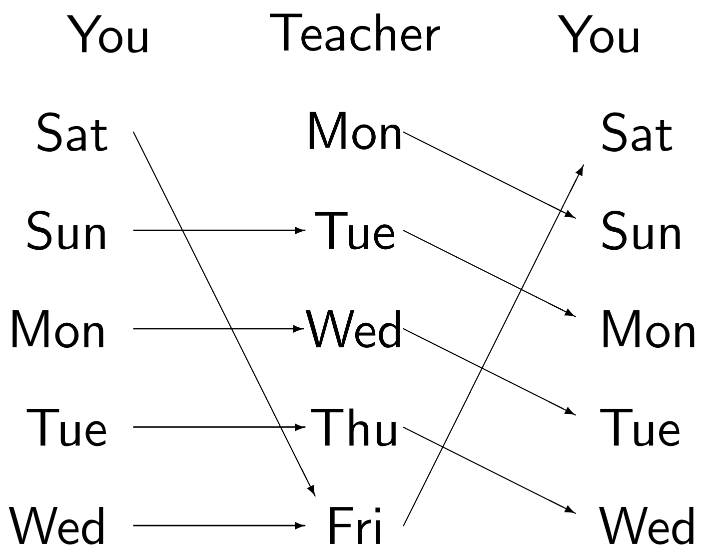
Utilities for teacher: Fail: $+5$, Studying one day: $+1$

Teacher

You

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

When should you start studying?



In a model with **standard beliefs,** every type expresses common belief in rationality.

So, under **common belief in rationality with standard beliefs** you can start studying on **any day.**

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

Now, turn to model with **lexicographic beliefs.**

If your type t_1 is **cautious,** Wed **cannot be rational** for you: Sat is always at least as good for you as Wed, and sometimes strictly better.

Say: Wed is **weakly dominated** for you by Sat.

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

Now, consider a type t_1 for you that expresses **common weak belief in rationality and cautiousness**.

Then, t_1 must weakly believe that teacher weakly believes that you will **not choose** Wed.

Reduced game

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$

Interpretation of reduced game: Your type t_1 weakly believes that teacher weakly believes that you will not choose Wed.

In reduced game, Thu is **strictly dominated** for teacher by Fri.

So, your type t_1 must weakly believe that teacher will **not choose** Thu.

Reduced game

	Mon	Tue	Wed	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$2, 3$

In reduced game, Tue is **strictly dominated** for you by Sat.

So, your type t_1 must weakly believe that teacher weakly believes that you will not choose Tue.

Reduced game

	Mon	Tue	Wed	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$1, 4$

In reduced game, Wed is **strictly dominated** for teacher by Fri.

So, your type t_1 must weakly believe that teacher will not choose Wed.

Reduced game

	Mon	Tue	Fri
Sat	$3, 2$	$2, 3$	$3, 6$
Sun	$-1, 6$	$3, 2$	$0, 5$
Mon	$0, 5$	$-1, 6$	$1, 4$

In reduced game, Mon is **strictly dominated** for you by Sat.

So, your type t_1 must weakly believe that teacher weakly believes that you will not choose Mon.

Reduced game

	Mon	Tue	Fri
Sat	$3,2$	$2,3$	$3,6$
Sun	$-1,6$	$3,2$	$0,5$

In reduced game, Tue is **strictly dominated** for teacher by Fri.

So, your type t_1 must weakly believe that teacher will not choose Tue.

Reduced game

	Mon	Fri
Sat	$3, 2$	$3, 6$
Sun	$-1, 6$	$0, 5$

In reduced game, Sun is **strictly dominated** for you by Sat.

So, your type t_1 can only rationally choose Sat.

So, if your type t_1 is **cautious**, **expresses common weak belief in cautiousness**, and **expresses common weak belief in rationality**, then the only rational choice for t_1 is to start studying on Sat.

We used the following **algorithm:**

First eliminate all **weakly** dominated choices.

Then apply iterated elimination of **strictly** dominated choices.

4.9 Algorithm

Consider a finite static game $\Gamma = (C_i, u_i)_{i \in I}$ with two players.

Say that choice c_i is **weakly dominated** if there is some randomized choice $\mu_i \in \Delta(C_i)$ with

$u_i(c_i, c_j) \leq u_i(\mu_i, c_j)$ for every $c_j \in C_j$, and

$u_i(c_i, c_j) < u_i(\mu_i, c_j)$ for at least some $c_j \in C_j$.

Algorithm: Dekel-Fudenberg Procedure

$$
\begin{aligned}
C_i^1 &:= \{c_i \in C_i \mid c_i \text{ not } \textbf{weakly} \text{ dominated}\} \\
C_i^2 &:= \{c_i \in C_i^1 \mid c_i \text{ not } \textbf{strictly} \text{ dominated on } C_j^1\} \\
C_i^3 &:= \{c_i \in C_i^2 \mid c_i \text{ not } \textbf{strictly} \text{ dominated on } C_j^2\} \\
&\;\;\vdots \\
C_i^k &:= \{c_i \in C_i^{k-1} \mid c_i \text{ not } \textbf{strictly} \text{ dominated on } C_j^{k-1}\} \\
&\;\;\vdots
\end{aligned}
$$

4.10 Why the algorithm works

Theorem 4.1: (Based on Brandenburger (1992), Börgers (1994))

A choice c_i can rationally be chosen by a **cautious** type that expresses **common weak belief in cautiousness and rationality**

if and only if

choice c_i survives the **Dekel-Fudenberg procedure.**

In order to prove this theorem, we need the following two results.

Lemma 4.2:

A choice c_i is optimal for some **cautious** lexicographic belief b_i on C_j

if and only if

c_i is **not weakly dominated.**

Lemma 4.3:

Let $D_j \subseteq C_j$ be a subset of opponent's choices.

A choice c_i is optimal for some **cautious** lexicographic belief b_i on C_j with $b_i^1 \in \Delta(D_j)$

if and only if

c_i is **not weakly dominated** on C_j and **not strictly dominated on D_j.**

Proof of Theorem 4.1.

Step 1. Take a cautious type t_i that expresses common weak belief in cautiousness and rationality.

Show: Type t_i must make a choice that survives the Dekel-Fudenberg procedure.

Define

$$C_i^1 := \{c_i \in C_i \mid c_i \text{ not } \textbf{weakly} \text{ dominated on } C_j \}.$$

Since t_i is cautious, we have by Lemma 4.2 that t_i must choose from C_i^1.

Define

$$C_i^2 := \{c_i \in C_i^1 \mid c_i \text{ not } \textbf{strictly} \text{ dominated on } C_j^1 \}.$$

Since t_i weakly believes in j's cautiousness and rationality, t_i must weakly believe that j chooses from C_j^1.

So, by Lemma 4.3, t_i must choose from C_i^2.

Define

$$C_i^3 := \{c_i \in C_i^2 \mid c_i \text{ not \textbf{strictly} dominated on } C_j^2 \}.$$

Since t_i weakly believes in j's cautiousness and rationality, and weakly believes that j weakly believes in i's cautiousness and rationality, t_i must weakly believe that j chooses from C_j^2.

So, by Lemma 4.3, t_i must choose from C_i^3.

And so on.

Hence, t_i must make a choice that survives the Dekel-Fudenberg procedure.

Step 2. Show that all choices surviving the Dekel-Fudenberg procedure can be chosen rationally by a cautious type that expresses common weak belief in cautiousness and rationality.

By applying the **Dekel-Fudenberg procedure,** we are left with subsets of choices D_1 and D_2 such that:

- every choice $c_i \in D_i$ is not weakly dominated in the original game, and

- every choice $c_i \in D_i$ is not strictly dominated on D_j.

So, by Lemma 4.3, every choice $c_i \in D_i$ can rationally be chosen under a lexicographic belief $b_i(c_i) = (b_i^1(c_i), ..., b_i^K(c_i))$ about j's choice where:

- $b_i(c_i)$ is **cautious,** and

- $b_i^1(c_i) \in \Delta(D_j)$.

Define sets of types $T_1 = \{t_1^{c_1} \mid c_1 \in D_1\}$ and $T_2 = \{t_2^{c_2} \mid c_2 \in D_2\}$ such that every type $t_i^{c_i}$ is **cautious**, and

$$b_i^1(t_i^{c_i})(c_j, t_j) = \begin{cases} b_i^1(c_i)(c_j), \text{ if } c_j \in D_j \text{ and } t_j = t_j^{c_j} \\ 0, \text{ otherwise} \end{cases}.$$

Then, every $t_i^{c_i} \in T_i$ expresses **common weak belief in cautiousness and rationality.**

Hence, for every $c_i \in D_i$ there is a type $t_i^{c_i}$ such that:

- t_i is cautious, and expresses common weak belief in cautiousness and rationality,

- c_i is rational for t_i.

So, every choice c_i surviving the Dekel-Fudenberg procedure can rationally be chosen by a cautious type t_i that expresses common weak belief in cautiousness and rationality.

This completes the proof of Step 2.

In particular, **common weak belief in cautiousness and rationality is always possible!**

4.11 Related Models

Selten (1975) presents a non-epistemic model that is based on **cautious reasoning** by players.

He models cautiousness by taking for player i a sequence $(\mu_j^n)_{n \in \mathbf{N}}$ of probability distributions on j's choice, where μ_j^n **assigns positive probability to each of** j's **choices.**

Informally, a **cautious lexicographic belief** about j's choice is "**equivalent**" to the limiting behavior of such a sequence $(\mu_j^n)_{n \in \mathbf{N}}$.

Dekel and Fudenberg (1990) study a model in which players have small doubts about opponent's utility function.

They show:

If one applies **iterated weak dominance**, and **uncertainty** about opponent's utility function **tends to zero**, then one obtains **Dekel-Fudenberg procedure**.

4.12 References

Lawrence Blume, Adam Brandenburger and Eddie Dekel (1991): "Lexicographic probabilities and choice under uncertainty", Econometrica 59, 61-79.

Adam Brandenburger (1992): "Lexicographic probabilities and iterated admissibility", in Economic Analysis of Markets and Games, Editors: Partha Dasgupta, Douglas Gale, Oliver Hart and Eric Maskin, The MIT Press, Cambridge, MA.

Tilman Börgers (1994): "Weak dominance and approximate common knowledge", Journal of Economic Theory 64, 265-276.

Eddie Dekel and Drew Fudenberg (1990): "Rational behavior with payoff uncertainty", Journal of Economic Theory 52, 243-267.

Reinhard Selten (1975): "Reexamination of the perfectness concept for equilibrium points in extensive games", International Journal of Game Theory 4, 25-55.