An Epistemic Course in Game Theory
Exercises to Lecture 5: "Common initial belief in rationality"

Problem 5.1: Common initial belief in rationality and backward induction.

Consider the following dynamic game between player 1 and player 2: First, player 1 makes a choice from $C = \{c_1, ..., c_K\}$. Player 2 observes the choice made by player 1, and subsequently makes a choice from $D = \{d_1, ..., d_L\}$. Then, the game ends. Clearly, this is a dynamic game with perfect information. Suppose that the game is free of ties.

(a) How many strategies does player 1 have? And player 2?

This game can be "solved" by using *backward induction:* First, after every possible choice c for player 1, you determine the optimal choice $d^*(c)$ for player 2. Given these optimal choices $d^*(c)$, you then determine the optimal choice c^* for player 1.

(b) Show that backward induction gives you a unique strategy s_1^* for player 1 and a unique strategy s_2^* for player 2. We call these strategies the *backward induction strategies* for the players.

(c) Show, by logical reasoning, that s_1^* and s_2^* are the only strategies that can rationally be chosen under common initial belief in rationality.

Suppose now that the game is extended as follows: After player 2 has made his choice, player 1 observes the choice made by player 2, and makes a choice from $E = \{e_1, ..., e_M\}$. Assume again the resulting game is free of ties.

(d) How many strategies does player 1 have in this new game?

Also this game can be "solved" by backward induction: After all possible choice combinations $(c, d) \in C \times D$ in the first two rounds, determine the optimal choice $e^*(c, d)$ for player 1 in the last round. Given these optimal choices, determine for every $c \in C$ the optimal choice $d^*(c)$ for player 2. Finally, given these optimal choices, determine the optimal choice c^* for player 1 in the first round. Again, this will result in unique backward induction strategies s_1^* and s_2^* for the players.

(e) Explain, intuitively, why common initial belief in rationality does not necessarily lead to the backward induction strategies in this new game.

Problem 5.2: A game of numbers.

Consider the following game between player 1 and player 2: First, player 1 chooses a number $a \in \{-1, 1\}$. Then, player 2 observes player 1's choice, and chooses a number $b \in \{-2, 1\}$. Finally, player 1 observes player 2's choice, and chooses a number $c \in \{-2, 3\}$. At the end, player 2 must pay the amount abc to player 1. (If abc is negative, this means that player 1 pays $|abc|$ to player 2).

(a) Model this situation as a formal dynamic game. Does the game have perfect information? Is the game free of ties?

(b) Apply backward induction to this game. What is the backward induction strategy s_1^* for player 1? What is the backward induction strategy s_2^* for player 2?

(c) Construct an epistemic model (T_1, T_2, b_1, b_2) with conditional beliefs such that:

- for player 1 there is a type $t_1^* \in T_1$ that expresses common initial belief in rationality, and for which strategy s_1^* is rational, and
- for player 2 there is a type $t_2^* \in T_2$ that expresses common initial belief in rationality, and for which strategy s_2^* is rational.

Hence, the backward induction strategies can rationally be chosen under common initial belief in rationality.

(d) Construct another epistemic model (T_1, T_2, b_1, b_2) with conditional beliefs such that:

- for player 1 there is a type $t_1^* \in T_1$ that expresses common initial belief in rationality, and for which a strategy $s_1 \neq s_1^*$ is rational, and
- for player 2 there is a type $t_2^* \in T_2$ that expresses common initial belief in rationality, and for which a strategy $s_2 \neq s_2^*$ is rational.

Hence, common initial belief in rationality does not necessarily lead to the backward induction strategies.

(e) Use the Ben-Porath procedure to find all strategies that can rationally be chosen under common initial belief in rationality.