

An Epistemic Course in Game Theory
Exercises to Lecture 3: “Common weak belief in rationality”

Practical problems

Problem 3.1: Who gets the task?

A colleague at work wants to paint his living room, but he has two left hands. Therefore, he wants to ask somebody else to do the job. Both you and your friend Barbara have told him to be interested in the task. In order to decide which person will get the job, he invites you and Barbara to his favorite pub, where he proposes the following procedure: Barbara and you must simultaneously write down a price on a piece of paper. The price must be either 100 euros, or 200 euros, or 300 euros. The person who has written down the lowest price will get the job, and will be paid exactly that amount. In case you both write down the same price, your colleague will choose Barbara, since she knows the colleague longer than you do.

- (a) Formulate this situation as a game between Barbara and you.
- (b) Which prices can you rationally write down under common belief in rationality with standard beliefs?
- (c) Now, turn to a setting with lexicographic beliefs. Which prices can you rationally write down if you have a cautious lexicographic belief, and express common weak belief in cautiousness and rationality?
- (d) Construct an epistemic model with lexicographic beliefs such that
 - every type in this model is cautious, and expresses common weak belief in cautiousness and rationality, and
 - for each of the prices found in (c) there is a type in this model for which this price is rational.

Problem 3.2: How to divide a cake?

It is your partner's birthday, and your mother-in-law has baked a delicious birthday cake. Experience has learned that dividing a cake between you and your partner usually results in a huge fight. In order to avoid this, you have agreed on the following procedure: First, you and your partner may claim a part of the cake, which can be any percentage between 0% and 100%, but it must be a whole number. However, you and your partner can also decide to leave the choice to the other person. If you have claimed a part x and your partner has claimed a part y with $x + y \leq 100$, then both claims are feasible, and each person will get the share he or she claimed. If $x + y > 100$, then together you have claimed too much and you will both get nothing. If you leave the choice to your partner, and your partner claims y , then your partner gets y and you will get the rest, which is $100 - y$. Similarly, if you claim a part x and your partner leaves the choice to you, then you get your share x and your partner gets the remaining part $100 - x$. If you both decide to leave the choice to the other person, then you are not able to reach a decision yourselves. In this case, your partner will call your mother-in-law, and she will then give 60% of the cake to your partner, and only 40% to you. Your final utility will be simply the percentage of the cake you eventually receive, and similarly for your partner.

(a) Show that claiming 40% of the cake or less is weakly dominated for you by some other choice. Which choice weakly dominates it?

(b) For each of your other choices c_1 , find a *cautious* lexicographic belief about the partner's choice for which c_1 is rational. Conclude that none of your other choices is weakly dominated.

(c) Similarly, show that for your partner claiming 60% of the cake or less is weakly dominated.

(d) For each of the other choices c_2 for your partner, find a *cautious* lexicographic belief about your own choice for which c_2 is rational. Conclude that none of your partner's other choices is weakly dominated.

(e) Consider the reduced game in which we have eliminated all weakly dominated choices. Show that, within the reduced game, claiming strictly more than 40% but strictly less than 100% is strictly dominated for you.

Here is a hint: Take a claim x between 41% and 99%. Show that, within the reduced game, claiming x is strictly dominated by a randomized choice in which you claim 100% with probability p , and in which you leave the choice to your partner with probability $1 - p$.

(f) Similarly, show that, within the reduced game, it is strictly dominated for your partner to claim strictly more than 60% but strictly less than 100%.

(g) Which choices for you and your partner survive the Dekel-Fudenberg procedure?

(h) Construct an epistemic model with lexicographic beliefs such that

- every type in this model is cautious, and expresses common weak belief in cautiousness and rationality, and
- for each of the choices found in (g) there is a type in this model for which this choice is rational.

Theoretical problem

Problem 3.3: Weak self-admissible sets.

Consider a finite, static two-player game with players i and j . Let $D_i \subseteq C_i$ and $D_j \subseteq C_j$ be subsets of the players' choices. Say that the pair (D_i, D_j) of choice subsets is a *weak self-admissible set*

- every $c_i \in D_i$ is not weakly dominated on C_j and not strictly dominated on D_j , and
- every $c_j \in D_j$ is not weakly dominated on C_i and not strictly dominated on D_i .

Show the following result: A choice $c_i \in C_i$ can rationally be chosen by some cautious type t_i that expresses common weak belief in cautiousness and rationality *if and only if* there is a weak self-admissible set (D_i, D_j) with $c_i \in D_i$.