An Epistemic Course in Game Theory
Exercises to Lecture 2: "Epistemic foundation for Nash equilibrium"

Practical problems

Problem 2.1: The lazy professor.

Consider the example "The lazy professor" we discussed in the lecture. Show that this game has a unique Nash equilibrium, namely

$$(\frac{1}{2}\{1,2,3\} + \frac{1}{2}\{1,2,4\}, \frac{1}{2}Ch.3 + \frac{1}{2}Ch.4).$$

Conclude that there are only two Nash choices for you, namely to study $\{1,2,3\}$ or to study $\{1,2,4\}$.

Here is a hint:
1. First show that in any Nash equilibrium (μ_1, μ_2), the probability distribution μ_2 must assign probability 0 to the professor's choice $Ch.1$. Here, you should try to use the fact that if μ_2 assigns positive probability to both $Ch.1$ and $Ch.2.$, then $Ch.1$ and $Ch.2.$ must give the professor the same expected utility against μ_1.
2. In a similar way, show that μ_2 must assign probability 0 to the professor's choice $Ch.2$.
3. Conclude that in a Nash equilibrium, it can never be optimal for you to choose $\{1\}, \{2\}$ or $\{1,2\}$, so μ_1 must assign probability 0 to $\{1\}, \{2\}$ and $\{1,2\}$.

Problem 2.2: Movie or party?

This evening you face a difficult decision. You have been invited to a party, together with your friends Barbara and Chris, but at the same time your favorite movie *Once upon a time in America* will be on TV. You prefer going to the party if you would have a good time there, but you prefer watching the movie rather than having a bad time at the party. More precisely, having a good time at the party gives you a utility of 3, watching the movie gives you a utility of 2, whereas having a bad time at the party gives you a utility of 0. Experience learns that you only have a good time at a party if both Barbara and Chris join. The problem, however, is that you are not sure whether both will come to the party, since they also like the movie a lot.
In fact, the decision problems for Barbara and Chris are similar to yours: Watching the movie gives them both a utility of 2, having a good time at the party gives them a utility of 3, whereas having a bad time at the party gives them a utility of 0. However, Barbara and Chris were having a fierce discussion yesterday evening, and for that reason would like to avoid each other today. More precisely, assume that Barbara will only have a good time at the party if you will be there but not Chris, and that Chris will only have a good time if you will be there but not Barbara.

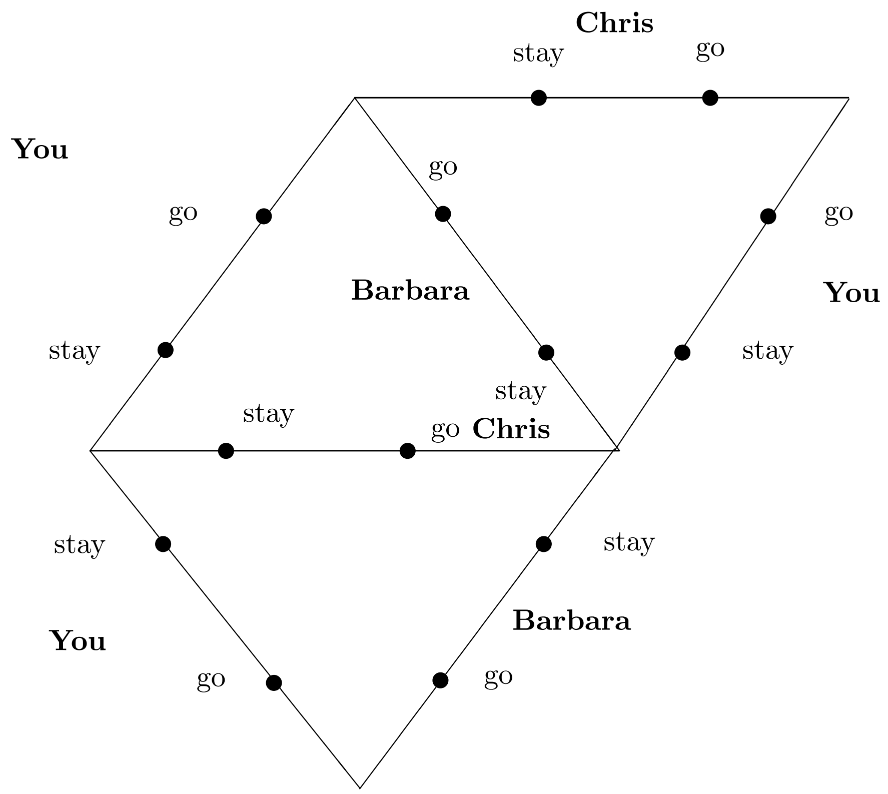
(a) Make a table like the one below, and fill in the utilities for the three players. Here, "stay" means you stay at home, whereas "go" means that you go to the party.

You stay	Chris stays	Chris goes
Barbara stays		
Barbara goes		

You go	Chris stays	Chris goes
Barbara stays		
Barbara goes		

(b) Make an extended beliefs diagram for this situation, and translate it into an epistemic model. Which of the types in the epistemic model have self-referential beliefs? Which types believe that the opponents have self-referential beliefs? Which types have projective beliefs? Which types have conditionally independent beliefs?

Hint: Since we have three players here, you could use the format on the next page to make the beliefs diagram. You must still fill in the arrows. Start at the middle triangle. There, you must support each of your rational choices by a belief about your friends' choices. For instance, it is optimal for you to "go" if you believe both your friends "go". So, draw an arrow from your choice "go" to Barbara's choice "go", and one from your choice "go" to Chris' choice "go". In the upper right triangle you must support each of Barbara's rational choices by a belief about her friends' choices. In the lower left triangle you must support each of Chris' rational choices by a belief about his friends' choices.



(c) Show that you can rationally choose to go to the party or to stay at home under common belief in rationality.

(d) Show that there is only one Nash equilibrium (μ_1, μ_2, μ_3) in this game, namely the one with $\mu_1(stay) = 1$, $\mu_2(stay) = 1$ and $\mu_3(stay) = 1$. Conclude from this that your only Nash choice is to stay at home.

Hint: Here is a way to prove it:
1. Suppose that $\mu_1(go) > 0$. Then, *go* must be optimal for you against (μ_2, μ_3). Show that this implies that $\mu_2(go) \geq \frac{2}{3}$ and $\mu_3(go) \geq \frac{2}{3}$.
2. Since $\mu_2(go) > 0$, *go* must be optimal for Barbara against (μ_1, μ_3). Show that this is not possible if $\mu_3(go) \geq \frac{2}{3}$.

3. Conclude from 1. and 2. that μ_1 cannot assign positive probability to *go*.
4. Show that there is only one Nash equilibrium.

Theoretical problems

Problem 2.3: Conditions in Theorem 3.1.

Consider a finite static game with two players, i and j. In Theorem 3.1, we considered the following four conditions on a type t_i:
A: Type t_i believes in j's rationality.
B: Type t_i believes that j believes in i's rationality.
C: Type t_i has self-referential beliefs.
D: Type t_i believes that j has self-referential beliefs.

In Theorem 3.1 we showed that, if type t_i satisfies conditions **A, B, C** and **D,** then every rational choice for t_i is a Nash choice.

(a) Show that none of the four conditions can be dropped in Theorem 3.1. So, suppose we would drop condition **A.** Then, construct a two-player game, an epistemic model and a type t_i such

- t_i satisfies conditions **B, C** and **D,** but not **A,**
- there is a rational choice c_i for t_i that is *not* a Nash choice.

Do the same for dropping conditions **B, C** and **D.**

(b) Let t_i be a type in a two-player game that satisfies the conditions **A, B, C** and **D** above. Show that t_i expresses common belief in rationality.

Problem 2.4. When does common belief in rationality lead to Nash choices?

Consider a finite static game with an arbitrary number of players. Suppose that every player i has a *unique* choice $\hat{c}_i$ that can be made rationally under common belief in rationality.

(a) Suppose we construct an epistemic model and a type $t_i \in T_i$ that expresses common belief in rationality. Show that t_i satisfies the seven conditions in Theorem 3.3.

(b) Use **(a)** to show that $\hat{c}_i$ is a Nash choice. So, if every player has a unique choice that can be made rationally under common belief in rationality, then all these choices will be Nash choices.

Now, consider a finite static game with two players, i and j, where both players have exactly two choices. Suppose that each of these four choices can be made rationally under common belief in rationality.

(c) Show that all choices in the game are Nash choices.