An Epistemic Course in Game Theory
Exercises to Lecture 1: "Common belief in rationality"

Practical problems

Problem 1.1: Preparing for a piano exam.

Suppose you study piano. Within two weeks you will have an important exam, but you have not been practicing too hard for it lately. There are three pieces that can possibly enter in the exam: a relatively easy piece by Mozart, a somewhat more difficult piece by Chopin, and a very tough piece by Rachmaninov. On the exam, the jury will select two out of these three pieces, but you do not know which ones. The jury will give you a grade for both pieces, and your final grade will be the average of these two grades. Since you have only two weeks left for practicing, you decide that you will focus on at most two pieces for the remaining two weeks. So, you can dedicate the full two weeks to a single piece, or you can dedicate one week to a piece and one week to another piece. Suppose that your expected grade for the Mozart piece is given by

$$4 + 3\sqrt{x},$$

where x is the number of weeks you dedicate to this piece. Similarly, your expected grade for the Chopin piece is

$$4 + 2.5x,$$

whereas the expected grade for the piece by Rachmaninov is given by

$$4 + 1.5x^2.$$

(a) Which are the rational practice schedules for you? For every rational practice schedule, find a belief about the jury's choice for which this schedule is optimal. For every irrational schedule, find a randomized choice that strictly dominates it.

Suppose now that the jury likes to see you perform well during the exam, but that they prefer listening to Chopin rather than listening to Rachmaninov, and they prefer Rachmaninov over Mozart. More precisely, the jury's utilities for listening to Chopin, Rachmaninov and Mozart are equal to 3, 2 and 1, respectively, and the jury's overall utility is given by the sum of your grade and the utilities they obtain from listening to the two pieces.

(b) Which selections of pieces are rational for the jury? For every rational selection, determine a belief about your practice schedule for which this selection is optimal. For every irrational selection, determine a randomized choice for the jury that strictly dominates it.

(c) Make an extended beliefs diagram for this situation, and translate it into an epistemic model.

(d) Within this epistemic model, consider your type that supports dedicating two weeks to Mozart, and your type that supports dedicating two weeks to Chopin. For both types, write down the complete belief hierarchy they have.

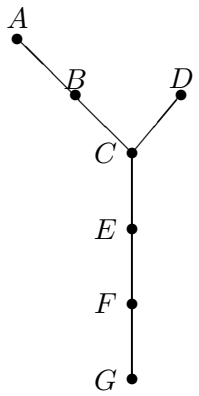
(e) Which of your types in this epistemic model believe in the jury's rationality? Which of these types believe, moreover, that the jury believes in your rationality? Which of your types express common belief in rationality?

(f) Which practice schedules can you rationally choose under common belief in rationality? For each of these practice schedules, construct an epistemic model and a type t_1 in this model that supports this choice, and expresses common belief in rationality. For each practice schedule *not* selected, explain why it cannot be chosen rationally under common belief in rationality.

(g) Use the algorithm of iterated strict dominance to find those practice schedules you can rationally choose under common belief in rationality, and compare your answer with (f). After how many rounds does the algorithm stop?

Problem 1.2: The mother-in-law.

Suppose, you and your partner are planning to move to another village, which consists of only three streets. There are seven houses for sale in that village, and their locations are depicted in the figure below.



The distance between two houses on this map is 100 metres. So, the distance between house A and house B is 100 metres, the distance between house B and house C is 100 metres, and so on.
When your mother-in-law learned about your plans, she decided to move to the same village as well, and she can choose from the same seven houses. Tomorrow, you and your mother-in-law have to sign up for one of the seven houses. If you both choose different houses, you will both get the house of your choice. If you both choose the same house, the house will go to a third party, and you will both not be able to move.
Suppose that your relationship with her is not quite optimal, and that you attempt to maximize the distance to her house. Your utility is equal to the distance between the houses if you both get a house in the village, and is equal to 0 if you cannot move to the village. The mother-in-law, on the other hand, wishes to minimize the distance to your house, as she likes to visit her child every day, and check whether you have cleaned the house properly. More precisely, the utility for your mother-in-law is equal to

$$600 - \text{ distance between the houses}$$

if you both get a house in the village, and is equal to 0 if she cannot move to the village.

(a) Show that location C is strictly dominated for you by a randomized choice in which you randomize over the locations A, D and G.

(b) Which are the rational locations for you? For every rational location, find a belief about your mother-in-law's choice for which this location is optimal. For every irrational location, find a randomized choice that strictly dominates it.

(c) Use the algorithm of iterated strict dominance to find those locations you can rationally choose under common belief in rationality. After how many steps does the algorithm stop?

(d) Which location, or locations, do you expect your mother-in-law to choose under common belief in rationality, and which not? Do you expect that both you and your mother-in-law could choose the same house?

(e) Construct an extended beliefs diagram such that each of your choices found in (c) is supported by a belief that only points towards your mother-in-law's choices found in (d), and vice versa.

(f) Translate this extended beliefs diagram into an epistemic model.

(g) For each of your choices found in (c), find a type for you in this epistemic model that expresses common belief in rationality and supports your choice. For each of these types, write down the belief hierarchy it has.

Problem 1.3: Going to a party.

Tonight you are going to a party with Barbara and Chris. You must decide which color to wear this evening. Suppose that you have the choice between white, red, blue, green, brown and black, and that your utilities for these colors are given by the following table.

white	red	blue	green	brown	black
3	6	2	5	1	4

Your friends Barbara and Chris can choose between the same colors. There is one problem, however: You do not know which colors they will wear this evening, and you *hate* wearing the same color as your friends. More precisely, if at least one of your friends will wear the same color as you do, your utility will be 0. If you wear a different color than both of your friends, your utility is given by the table.

(a) Which colors are rational for you? For each of the rational colors, find a belief about your friends' choices for which this color is optimal. For each of the irrational colors, find a randomized choice that strictly dominates it.

Suppose that Barbara and Chris hold other preferences over colors than you do. Their utilities are given by the following table.

	white	red	blue	green	brown	black
Barbara	4	1	6	2	3	5
Chris	6	4	3	1	5	2

Both Barbara and Chris hate wearing the same color as their friends. That is, if at least one of their friends wears the same color, their utility would be zero.

(b) Which colors are rational for Barbara and Chris?

(c) Make an extended beliefs diagram, and translate it into an epistemic model.

(d) Within the epistemic model, consider your type that supports wearing red, and your type that supports wearing white. For both types, write down the complete belief hierarchy they have.

(e) Which of your types in this epistemic model, if any, expresses common belief in rationality?

(f) Find those colors you can rationally choose under common belief in rationality. For each of those colors, construct an epistemic model and a type for you in this model that supports this choice and that expresses common belief in rationality. For every other color, argue why it cannot be chosen under common belief in rationality.

(g) Use the algorithm of iterated strict dominance to find those colors you can rationally choose under common belief in rationality, and compare your answer with (f).

Suppose now that the preferences for Chris are given by the following table.

	white	red	blue	green	brown	black
Chris	1	4	3	6	5	2

The preferences for you and Barbara remain unchanged. Answer questions (f) and (g) for this new situation.

Theoretical problem

Problem 1.4: Best-response sets.

Let $\Gamma = (C_i, u_i)_{i \in I}$ be a finite static game. Consider a profile $(D_i)_{i \in I}$ of subsets of choices, where $D_i \subseteq C_i$ for every player i. The profile $(D_i)_{i \in I}$ is called a *best-response set* if for every player i, every $c_i \in D_i$ is optimal for some belief in $\Delta(D_{-i})$. Here, $D_{-i} := \times_{j \neq i} D_j$.

Show the following result: A choice c_i can rationally be chosen under common belief in rationality *if and only if* there is a best-response set $(D_i)_{i \in I}$ with $c_i \in D_i$.