Weighted Nucleoli *

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Abstract

Cooperative games in characteristic function form (TU games) are considered. We allow for variable populations or carriers. Weighted nucleoli are defined via weighted excesses for coalitions. A solution satisfies the Null Player Out (NPO) property, if elimination of a null player does not affect the payoffs of the other players. For any single-valued and efficient solution, the NPO property implies the null player property. We show that a weighted nucleolus has the null player property if and only if the weights of multi-player coalitions are weakly decreasing with respect to coalition inclusion. Weighted nucleoli possessing the NPO-property can be characterized by means of a multiplicative formula for the weights of the multi-player coalitions and a restrictive condition on the weights of one-player coalitions.

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1 Introduction

The nucleolus as originally introduced by Schmeidler (1969) for cooperative games in characteristic function form (TU games) with a non-empty set of imputations, selects the unique imputation that successively (lexicographically) minimizes the maximal excesses. This defining property makes the nucleolus appealing as a "fair" single-valued solution. That outstanding feature becomes even more poignant in view of the fact that the nucleolus belongs to the core (resp. epsilon-core resp. least core), if the respective set is non-empty — and thus points to the location of the core and serves as a substitute for the core, when the core is empty. Let us add a few more properties that enhance the attractiveness of the nucleolus. First, the nucleolus is covariant, symmetric and exhibits the null player property: null players receive zero payoffs. But the nucleolus is non-linear or, more precisely, non-additive. However, it shares with many prominent linear values what we call the "null player out" property or NPO for short: i.e. if a null player is deleted from the game, the utility allocation of the remaining players is unaffected by his departure. Because of these several desirable properties, the nucleolus and nucleolus-like solution concepts have found plenty of applications in cost sharing, revenue distribution, resource allocation, assignment and matching problems, as Maschler (1992) reports. However, the nucleolus possesses certain features that render it less palatable in certain situations. For instance, the nucleolus lacks many monotonicity properties; see Young (1985) and Maschler (1992). In extreme examples, it prescribes implausible utility allocations. Above all, the original definition treats the excesses of any two coalitions as equally important – regardless of coalition sizes and coalition composition. The concept of a weighted nucleolus relaxes this not very compelling condition.

In this paper, we study weighted nucleoli. The formal definition of a weighted nucleolus coincides with the one for the standard nucleolus, except that excesses are replaced by weighted excesses. The weighted excesses are obtained by multiplying the ordinary excess of each coalition with a coalition specific positive coefficient or weight. Given any such weight system, existence and uniqueness of the weighted nucleolus are guaranteed. Moreover, two weighted nucleoli are equal on all eligible games if and only if the two involved weight systems are equal up to a positive multiplicative factor. In the proof of the latter result as well as of several other results, we rely on a “Kohlberg condition”, i.e. a set of balancedness conditions determining a weighted nucleolus.

Wallmeier (1983) studies weighted nucleoli where the weights depend on coalition size only and are weakly decreasing in coalition size. This property is satisfied by the standard nucleolus where each coalition has unit weight. It is also satisfied by the per capita nucleolus where each coalition is weighted with the reciprocal value of its size. Wallmeier offers two reasons why weights ought to be decreasing in coalition size which can be paraphrased as follows:

1) Consider a utility allocation and a small and a large coalition with the same positive excess with respect to the given allocation. Then on average, a member of the small coalition has more to gain from the formation of this coalition than a member of the large coalition can expect from the formation of that coalition.

2) Formation of small coalitions is simpler than formation of big ones.

These two arguments have merit, but warrant qualification, nonetheless. Concerning 1), one should not forget that politically and sociologically, not only the magnitude of individual dissatisfaction matters, but also the sheer number of dissatisfied people voicing their concerns. With respect to 2), it is doubtful that the weight of a coalition is the best descriptive tool to account for frictions in forming that coalition. Ideally, the worth of a coalition should reflect already the monetary or resource cost or more generally “transaction costs” required to form the coalition. Further impediments to coalition formation could be captured by means of communication structures, e.g.

Wallmeier (1983) shows in passing that his assumption of weights decreasing in coalition size implies the null player property for weighted nucleoli.[1] We establish in our context that a weighted nucleolus has the null player property if and only if the corresponding (possibly coalition specific) weight system, excluding the one-player coalition weights, is weakly decreasing with respect to coalition inclusion. Hence any argument in favor of the null player property lends some support to Wallmeier’s assumption and vice versa.

First and foremost, our curiosity about the NPO-property has been inspired by comparisons of two types of collusion between two players: fusion or amalgamation à la Lehrer (1988) and a proxy or representation agreement à la Haller (1994). In a proxy agreement, both players continue to exist physically; one of them becomes a null player, whereas the proxy player gets

[1] See Wallmeier (1983), Satz 8.7(ii). Observe that weighted nuceoli are covariant.

the power to act and sign on behalf of both of them. Amalgamation means that the number of players is reduced; the two colluding players are replaced by a single player who acts and signs on behalf of both of them. Technically, the case of amalgamation differs from the proxy case in that in the former type of collusion the designated null player is removed. Haller (1994) demonstrates that this seemingly minor difference can be crucial. More precisely, he considers (in the standard sense) symmetric probabilistic solutions and the effect of collusion on the sum of the solution outcomes of the two colluding players. He finds that the difference between proxy agreement and amalgamation does not matter if and only if a reduction axiom equivalent to the NPO-property is satisfied.

After characterizing weighted nucleoli with the null player property, we determine the class of weighted nucleoli satisfying NPO[2]. This concept presupposes variable player sets. We observe that NPO plus efficiency imply the null player property. Since by definition weighted nucleoli are efficient, NPO imposes more restrictions on the weight system than those required for the null player property. It turns out that weighted nucleoli with the NPO-property are characterized by a multiplicative decomposition of the weight system. Indeed, the latter property implies weights decreasing with respect to coalition inclusion. A multiplicative weight structure is trivially given for the standard nucleolus, but is lacking for the per capita nucleolus. Therefore the former possesses and the latter lacks NPO.

To reiterate, the main purpose of this investigation is to characterize weighted nucleoli with the null player property and the null player out property, respectively. An important tool for our analysis is a Kohlberg condition for weighted nucleoli which is a direct generalization of the one for the standard nucleolus introduced in Kohlberg (1971).

In the concluding section we summarize our findings, and report briefly on several nucleolus-like solution concepts that have been suggested recently.

[2]In a companion paper, Derks and Haller (1995), results on linear solutions satisfying NPO are collected.

2 Preliminaries

We consider a *universal player set* $\mathcal{N}$ with generic elements $i, j, \ldots$. To exclude trivial cases we assume that $\mathcal{N}$ consists of at least 4 players. Let $\mathcal{F}$ denote the set of finite subsets N of $\mathcal{N}$. To each $N \in \mathcal{F}$, we associate $\mathcal{G}^N$, the set of cooperative transferable utility games (N, v), with player set N and characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. $\mathcal{G}$ denotes the *universal game space* $\bigcup_{N \in \mathcal{F}} \mathcal{G}^N$.

The subsets of the player set are called the coalitions, and the worth $v(S)$ of a coalition S in the game (N, v) represents the profit the players in S may encounter in case they are in full cooperation with each other. We allow in our context that players may enter or leave the game, and we denote the game (M, w), with $M \in \mathcal{F}$ and defined by $w(S) = v(N \cap S)$, $S \subseteq M$, simply by (M, v).

For $(N, v) \in \mathcal{G}$, let $I(N, v)$ denote the set of allocations $x \in \mathbb{R}^N$, which are *individual rational*, $x_i \geq v(\{i\})$ for all $i \in N$, and *efficient*, $x(N) = v(N)$ (where $x(S)$ denotes the summation $\sum_{i \in S} x_i$, with the convention that $x(\emptyset) = 0$). $I(N, v)$ is called the imputation set, and its elements *imputations*. Let $\mathcal{I}$ denote the set of games in $\mathcal{G}$ with non-empty imputation set.

A solution concept (solution for short) is a single-valued function on (a subset of) the universal game space. If N is the player set the solution takes values from the *allocation space* $\mathbb{R}^N$. A solution is called efficient if it allocates the worth of the grand coalition, the player set, among the players.

A player i is called a *null player* in the game (N, v) if $v(S \cup \{i\}) = v(S)$ for each $S \subseteq N/\{i\}$, i.e., a null player does not influence the worth of a coalition he enters. A solution ϕ is said to have the *null player property* if $\phi_i(N, v) = 0$ whenever i is a null player in (N, v). We are now especially interested in solutions ϕ with the property

$$\phi_i(N, v) = \phi_i(N/\{j\}, v) \tag{1}$$

for all $i, j \in N$ such that j is a null player in (N, v). We call this the *"null player out" property* or **NPO** for short.

We adopt the following notational conventions. For $j \in \mathcal{N}$, $S \in \mathcal{F}$, S/j and $S \cup j$ are shorthand for $S/\{j\}$ and $S \cup \{j\}$; $\subseteq$ denotes weak set inclusion, whereas $\subset$ denotes strict (proper) set inclusion. We distinguish between one-player coalitions and multi-player coalitions.

3 Weighted Nucleoli

Given weights $p_S^N > 0$ for $\emptyset \subset S \subset N$, and $N \in \mathcal{F}$, a game $(N, v) \in \mathcal{I}$, and imputation $x \in I(N, v)$, and a coalition $S \subseteq N$, we define the *weighted excess*

$$e^p(N, v; S, x) \equiv p_S^N(v(S) - x(S)).$$

The superscript p refers to the weight system $p = (p_S^N)$. The weighted excess is the weighted gap between the profit and the accumulated payoff to the players in the coalition, and as such it may be interpreted as the dissatisfaction of the coalition towards the proposed payoff. It should be noted that the adjustment, by applying the weight structure, only depends on the coalition and the player set, and not on the coalition worth or the imputation. The characterizations in the sequel are essentially based upon this feature of the weight system.

Consider the 2^n-dimensional vector $\theta^p(N, v; x)$, with $n = |N|$, whose components are the weighted excesses of the 2^n subsets S of N, arranged in non-increasing order. Let $\leq_L$ denote the lexicographic order on $\mathbb{R}^{2^n}$, and set

$$\nu^p(N, v) \equiv \{x \in I(N, v) : \theta^p(N, v; x) \leq_L \theta^p(N, v; y) \text{ for all } y \in I(N, v)\}. \tag{2}$$

In complete analogy to the standard methods for investigating the nucleolus [see, e.g. Owen (1982), Section XI.3], one can demonstrate that the set $\nu^p(N, v)$ is a singleton.[3] Its single element is also denoted $\nu^p(N, v)$ and called the *weighted nucleolus* of the game (N, v) with defining weight system p. The *(standard) nucleolus* corresponds to the weight system $p_S^N = 1$ for all coalitions S and player sets N. The *per capita nucleolus* is given by the weight system $p_S^N = 1/|S|$. Like the nucleolus, a weighted nucleolus $\nu^p(N, v)$ belongs to the core of (N, v), if the core is non-empty, where the *core* of a game is defined as the set of imputations with non-positive excess for each coalition. The core is a convex and compact subset of the allocation space. It is proved in Yanovskaya (1992) that each allocation in the relative interior of the core may appear as the weighted nucleolus of the game, with weights chosen appropriately.

Our analysis reveals that not all properties of the nucleolus are inherited by an arbitrary weighted nucleolus. We first address the question to what

[3]This has been asserted before by Justmann (1977), among others. See also Derks and Peters (1998), where alternatives for the lexicographic order are discussed.

extent two different weight systems determine identical weighted nucleoli. The answer is given by

Theorem 1 *Two weighted nucleoli are equal on all games in the set I^N of games with player set N and with non-empty imputation set, if and only if the two involved weight systems, restricted to the player set N, are equal up to a positive multiplication factor.*

In the proof of this theorem as well as of several subsequent results, we rely on the following characterization of the weighted nucleoli by means of a set of balancedness conditions.

Given (N, v), $x \in I(N, v)$, and weight system $p = (p_S^N)$, consider the following collection of coalitions

$$\begin{aligned} E^p(N, v; t, x) &= \{S \subseteq N : \ e^p(N, v; S, x) \geq t\}, \ t \in \mathbb{R}, \text{ and} \\ T(N, v; x) &= \{i \in N : \ x_i > v(\{i\})\}. \end{aligned}$$

The set $E^p(N, v; t, x)$ consists of the coalitions with weighted excess at least the level t; the set $T(N, v; x)$ consists of the players each of whose allocation in x strictly exceeds his individual worth in the game (N, v).

For a given $T \subseteq N$ we say that a collection $B \subseteq 2^N$ of coalitions is T-*balanced* if weights $\lambda_S > 0$, $S \in B$, exist such that $\sum_{S \in B, i \in S} \lambda_S$ equals 1 for the players i in T, and is at most 1 for the players outside T. N-balancedness amounts to the standard notion of balancedness, and we sometimes omit the prefix "N-" in this case. Observe that balancedness implies T-balancedness for each $T \subseteq N$. An empty set is assumed to be balanced.

Theorem 2 (The Kohlberg Condition) *A necessary and sufficient condition for $x \in I(N, v)$ to be the weighted nucleolus $\nu^p(N, v)$ is that $E^p(N, v; t, x)$ is $T(N, v; x)$-balanced for each level t.*

For a proof we refer to Potters and Tijs (1992). It has been considered first in Kohlberg (1971) for the standard nucleolus.

The following lemma describes a result which will be quite useful for our analysis. Here, a collection is called *minimal balanced* if it is balanced and does not contain a balanced collection as a proper subset.

Lemma 1 *Let p be a weight system on $2^N/\{\emptyset, N\}$. Then p is constant if it is constant on each minimal balanced collection. Further, p is constant on the multi-player subsets if it is constant on each minimal balanced collection containing only multi-player sets.*

Proof: For a non-empty subset S of N define the collection $B^S = \{S\} \cup \{N/i : i \in S\}$. It is evident that B^S is minimal balanced, with balancing weights all equal to $1/|S|$. The assumption is that p is constant on B^S, i.e., $p_{N/i} = p_S$ for all $i \in S$. By the same reasoning, $p_{N/i} = p_T$ for each $i \in T$, and $T \subset N$ arbitrary. This shows that $p_S = p_T$, and, thus, p is constant on $2^N/\{\emptyset, N\}$.

Observe that B^S consists of multi-player sets whenever S is a multi-player coalition. Thus, the second assertion in the lemma also holds. □

Proof of Theorem 1: The "if"-part of the theorem is evident. To derive the "only if"-part, let us consider two weighted nucleoli ν^p and ν^q such that $\nu^p(N, v) = \nu^q(N, v)$ for all $(N, v) \in \mathcal{I}^N$. Next consider a minimal balanced collection B, with $B \neq \{N\}$, and, further, consider the game (N, w) :

$$w(S) = \begin{cases} 0 & \text{if } S = \emptyset \text{ or } S = N; \\ -1/p_S^N & \text{if } S \in B; \\ -K/p_S^N & \text{otherwise,} \end{cases}$$

with $K > 1$ chosen sufficiently large. The zero-allocation 0 is an imputation for this game, with p-weighted excesses 0 for the coalitions $\emptyset$ and N, -1 for those in B, and $-K$ otherwise. This shows that 0 is the p-weighted nucleolus of (N, w). The excesses of the allocation 0 wrt. the weight system q are

$$e^q(N, w; S, 0) = \begin{cases} 0 & \text{if } S = \emptyset \text{ or } S = N; \\ -q_S^N/p_S^N & \text{if } S \in B; \\ -Kq_S^N/p_S^N & \text{otherwise.} \end{cases}$$

Let K be such that $-q_S^N/p_S^N > -Kq_U^N/p_U^N$ for all $S \in B$, and $U \notin B \cup \{\emptyset, N\}$. Then, with $t = -q_T^N/p_T^N$ for a $T \in B$, we have

$$\{\emptyset, T, N\} \subseteq E^q(N, w; t, 0) \subseteq B \cup \{\emptyset, N\}.$$

¿From $0 = \nu^p(N, w) = \nu^q(N, w)$ and $w(\{i\}) < 0$, $i \in N$, we conclude that $E^q(N, w; t, 0)$ has to be N-balanced, and this is only possible if $E^q(N, w; t, 0) = B \cup \{\emptyset, N\}$. Since the latter has to hold for every $T \in B$, we conclude that

$$\frac{q_S^N}{p_S^N} = -t \quad \text{for all } S \in B.$$

The collection B is chosen arbitrarily among the minimal balanced collections. Applying Lemma 1, we may therefore conclude that $q_S^N/p_S^N = -t$ for all non-empty $S \subset N$. This completes the proof of Theorem 1.

4 Main Results

Since they are imputations, weighted nucleoli are efficient. Together with NPO this obviously implies the null player property. Therefore, the first step towards a characterization of weighted nucleoli with the NPO-property is the characterization of weighted nucleoli with the null player property. In this section we will present both characterizations. These results show a stark distinction between the one-player and the multi-player coalition weights. This is due to the fact that imputations have non-positive excesses for the one-player coalitions, while the weighted excesses of the multi-player coalitions can be positive and, therefore, may have a larger influence on the weighted nucleolus.

4.1 The Null Player Property

The characterization of weighted nucleoli with the null player property, is presented in the next theorem.

Theorem 3 *The weighted nucleolus ν^p has the null player property on the set of games with player set $N \in \mathcal{F}$ if and only if*

$$p_S^N \geq p_T^N \quad \text{for all } S \subset T \subset N,\ |S| > 1. \tag{3}$$

Proof: **Proof of "if":** Let (N, v) be a game with null player $j \in N$. If the core is non-empty, then $\nu^p(N, v)$ is a core allocation. Each core allocation x must fulfil $x_j = 0$, and therefore $(\nu^p(N, v))_j = 0$.

Now suppose the core of (N, v) is empty. We will show that for any imputation x with $x_j > v(\{j\})$, there exists an imputation y with smaller maximal excess. The emptiness of the core implies that the maximal excess of the imputation x is positive. Let S denote a coalition whose weighted excess $e^p(N, v; S, x) = p_S(v(S) - x(S))$ is maximal. Observe that S is a multi-player coalition, since the excesses for the one-player coalitions are non-positive. We distinguish two cases:

- $j \in S$ for some coalition S with maximal weighted excess. Then S/j is also a multi-player coalition, since $v(S/j) - x(S/j) = v(S) - x(S) + x_j >$

0. Thus, $p_S^N \leq p_{S/j}^N$, and, therefore,

$$\begin{aligned} e^p(N,v;S,x) &= p_S(v(S)-x(S)) = p_S(v(S/j)-x(S/j)) - p_S x_j \\ &\leq p_{S/j}(v(S/j)-x(S/j)) - p_S x_j \\ &< e^p(N,v;S/j,x), \end{aligned}$$

contradicting the assumption that the excess at S is maximal.

- $j \notin S$ for each coalition S with maximal weighted excess. Consider the imputation y with coordinates $y_j = x_j - \epsilon$ and $y_i = x_i + \frac{\epsilon}{n-1} x_j$, for $i \in N/j$, where $1 > \epsilon > 0$ is chosen such that the weighted excesses $e^p(N,v;S,y)$ of the coalitions S with maximal weighted excess wrt. x are still greater than the excesses $e^p(N,v;T,y)$ of the other coalitions T. Observe that the maximal weighted excess wrt. imputation y is smaller than the maximal weighted excess wrt. x. This implies that x cannot be equal to $\nu^p(N,v)$.

We conclude that $(\nu^p(N,v))_j = v(\{j\}) = 0$.

Proof of "only if": Suppose that the weight system does not fulfil the conditions mentioned in the theorem. Then there exist a multi-player coalition S and a player $j \notin S$ such that $S \cup j \neq N$ and $p_S < p_{S \cup j}$. Consider the $\{0, 1/2, 1\}$-valued game v where the 1-valued coalitions are S and $S \cup j$, and the 1/2-valued coalitions are N/j and N. Player j is a null player in (N,v). The game has an empty core; its imputation set, however, is non-empty. Now let x be an imputation with $x_j = 0$. The maximal weighted excesses occur only at coalitions S and $S \cup j$. Since $x_j = 0$ and $p_S < p_{S \cup j}$, the weighted excess of $S \cup j$ is larger than the one of S. A small redistribution of the worth $v(N) = 1/2$ to player j reduces the weighted excess of $S \cup j$ without making it smaller than the weighted excess of any other coalition. This implies that x cannot be the weighted nucleolus. Therefore, $(\nu^p(N,v))_j > 0$. □

Remark. The **'if'**-part of the theorem encompasses Satz 8.7(ii) in Wallmeier (1983). Both parts of the foregoing proof rely on the individual rationality of imputations. Thus, the identical argument cannot be used, if individual rationality is dropped. This poses the question whether an analogue of the theorem holds true if the individual rationality requirement is dropped in the definition of the weighted nucleolus. Solutions evolving from this generalization are the so called weighted prenucleoli. The following example shows that weighted prenucleoli do not possess the null player property in situations where the weights are decreasing wrt. coalition inclusion.

Example. Consider the 3-person game v, defined in the next table, together with the excesses wrt. the allocation $x = (-1, 1, 1)$.

S:	1	2	3	12	13	23	123
p_S:	1	1	1	1/2	1/2	1/2	
$v(S)$:	0	2	2	2	2	1	1
$x(S)$:	-1	1	1	0	0	2	1
$e^p(N, v; S, x)$:	1	1	1	1	1	$-1/2$	0

The coalitions with maximal excess 1 form a balanced collection, and this is also the case for the other levels, showing that x is actually the prenucleolus. Further, observe that player 1 is a null player; his payoff in the prenucleolus is unequal to 0, i.e., the (weighted) prenucleolus does not fulfil the null player property.

4.2 The Null Player Out Property

NPO means that in a sense the weighted nucleoli $\nu^p(N \cup j, .)$ and $\nu^p(N, .)$ are the same, on the set of games where j is a null player. In that case Theorem 1 suggests that the weights $p_{\cdot}^{N \cup j}$ and $p_{\cdot}^{N}$ are identical or proportional. This intuition is partly confirmed by the following result.

Lemma 2 *Suppose NPO holds for the weighted nucleolus defined by the weight system $p = (p_S^N)$. Then for all $N \in \mathcal{F}, j \in \mathcal{N}/N$, there exist constants $d_{N,j} > 0$ such that*

$$p_S^{N \cup j} = d_{N,j} p_S^N \quad \textit{for all multi-player coalitions } S \subset N. \tag{4}$$

Proof: Suppose p defines a weighted nucleolus satisfying NPO. Consider an arbitrary minimal balanced collection $B \subset 2^N$, non-empty, unequal to $\{N\}$, and consisting only of multi-player coalitions. Let $j \notin N$ and consider the following game w on the player set $N \cup j$:

$$w(S) = \begin{cases} 1/p_{S/j}^{N \cup j} & \text{if } S/j \in B; \\ -1/p_{S/j}^{N \cup j} & \text{if } |S/j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that player j is a null player. The excesses wrt. the zero-allocation, which is an imputation, are

$$e^p(N \cup j, w; S, 0) = \begin{cases} 1 & \text{if } S \in B; \\ p_S^{N\cup j}/p_{S/j}^{N\cup j} & \text{if } S \ni j \text{ and } S/j \in B; \\ -1 & \text{if } |S| = 1 \text{ and } S \neq \{j\}; \\ -p_S^{N\cup j}/p_{S/j}^{N\cup j} & \text{if } |S| = 2 \text{ and } j \in S; \\ 0 & \text{otherwise.} \end{cases}$$

The null player property and Theorem 3 imply that the maximal excess is 1. From this we see that

$$B \subseteq E^p(N \cup j, w; t, 0) \subseteq B \cup \{S \cup j : \ S \in B\}$$

for excess levels $t : \ 1 \geq t > 0$. For non-positive levels t it is clear that

$$2^N/\{\{i\} : \ i \in N\} \subseteq E^p(N \cup j, w; t, 0).$$

In both cases one easily shows that $E^p(N \cup j, w; t, 0)$ has to be N-balanced. Applying Theorem 2 and $N = T(N \cup j, w; 0)$, we conclude that 0 is the weighted nucleolus of $(N \cup j, w)$. Thus, applying the NPO-assumption, the zero-allocation in $\mathbb{R}^N$ is the weighted nucleolus of (N, w). Consider its excesses

$$e^p(N, w; S, 0) = \begin{cases} p_S^N/p_S^{N\cup j} & \text{if } S \in B; \\ -p_S^N/p_S^{N\cup j} & \text{if } |S| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

With $t = p_T^N/p_T^{N\cup j}$ for a $T \in B$, we must have

$$\frac{p_S^N}{p_S^{N\cup j}} = t \quad \text{for all } S \in B.$$

Therefore, by Lemma 1, this identity has to hold for all coalitions S, $|S| > 1$. Letting $d_{N,j}$ be equal to $1/t$ we obtain the assertion in the lemma. □

Lemma 3 *Suppose NPO holds for the weighted nucleolus defined by the weight system $p = (p_S^N)$. Then for all $N \in \mathcal{F}, j \notin N$, there exist positive constants $c_{N,j}$ such that*

$$p_S^N = c_{N,j} \min(p_{S\cup j}^{N\cup j}, p_S^{N\cup j}) \quad \textit{for all non-empty } S \subset N. \tag{5}$$

Proof: Consider a non-empty, minimal balanced collection $B \subset 2^N$, different from $\{N\}$. Further, consider the game (N, w), defined by

$$w(S) = \begin{cases} 0 & \text{if } S = \emptyset \text{ or } S = N; \\ -1/p_S^N & \text{if } S \in B; \\ -K/p_S^N & \text{otherwise,} \end{cases}$$

with $K > 1$ sufficiently large. It is evident that $0 \in \mathbb{R}^N$ is the weighted nucleolus of this game. The game $(N \cup j, w)$, where player j is added to the above game as a null player, has also $0 \in \mathbb{R}^{N \cup j}$ as the weighted nucleolus. A closer look to the excesses will reveal the property mentioned in the lemma:

$$e^p(N \cup j, w; S, 0) = \begin{cases} 0 & \text{if } S = \emptyset, \{j\}, N, N \cup j; \\ -p_S^{N \cup j}/p_{S/j}^N & \text{if } S/j \in B; \\ -K p_S^{N \cup j}/p_{S/j}^N & \text{otherwise.} \end{cases}$$

The maximal excess is 0. Assume that K is chosen such that the next maximal excess, denoted by t, occurs only for coalitions S, with $S/j \in B$. Let B' be the set $\{S/j : S \in E^p(N \cup j, w; t, 0)\}$. Now $E^p(N \cup j, w; t, 0)$ is N-balanced according to the Kohlberg condition and the observation $w(\{i\}) < 0$ for $i \in N$. From this one easily shows the N-balancedness of B'. N-balancedness must also hold for B'/N which is non-empty; since B'/N is a subset of the minimal balanced collection B we conclude that $B'/N = B$, and therefore,

$$\max(-\frac{p_{S \cup j}^{N \cup j}}{p_S^N}, -\frac{p_S^{N \cup j})}{p_S^N}) = t \quad \text{for all } S \in B. \tag{6}$$

The minimal balanced collection B is chosen arbitrary so that we may conclude, with the help of Lemma 1, that (6) holds for all non-empty subsets of N. By choosing $c_{N,j}$ to be equal to $-1/t$ the assertion in the lemma follows. □

NPO implies the null player property, so that the weight system p in the above lemma has to fulfil the assertion in Theorem 1, i.e., $p_{S \cup j}^{N \cup j} \leq p_S^{N \cup j}$ for multi-player coalitions $S \subset N$. Together with (5) we obtain

$$p_S^N = c_{N,j} p_{S \cup j}^{N \cup j} \quad \text{for all multi-player } S \subset N.$$

Therefore, Lemma 3 implies the less evident assertion that NPO also requires proportionality of the weights for multi-player coalitions with and without the null player.

We are now in the position to state our main result.

Theorem 4 *The NPO-property holds for the weighted nucleolus defined by the weight system $p = (p_S^N)$ if and only if the null player property, (4) and (5) hold.*

Proof: The previous two lemmata show the "only if" case. The "if" case is proved as follows.

Consider an arbitrary player set $N \in \mathcal{F}$ and player j, not in N. Following Theorem 1 it is no loss of generality if we assume that the constant $c_{N,j}$ in (5) equals 1. Let d denote the constant $d_{N,j}$ of (4). So, $p_S^N = \min(p_{S\cup j}^{N\cup j}, p_S^{N\cup j})$, and in particular $p_S^N = p_{S\cup j}^{N\cup j}$ for multi-player S, and $p_S^{N\cup j} = dp_S^N$. It follows that $d \geq 1$.

Let (N, w) be an arbitrary game, and x an imputation of this game. Then $x' \in \mathbb{R}^{N\cup j}$, with $x_i' = x_i$ for $i \in N$, and $x_j' = 0$ is an imputation of $(N \cup j, w)$, where j is treated as a null player. Let T be the set of players $i \in N$ with $x_i > w(\{i\})$. We have to prove that all collections $E^p(N, w; t, x)$, $t \in \mathbb{R}$, are T-balanced if and only if all collections $E^p(N \cup j, w; t, x')$, $t \in \mathbb{R}$, are T-balanced.

Let B_t denote the set $E^p(N, w; t, x)$, and D_t denote the set $\{S/j : S \in E^p(N \cup j, w; t, x')\}$. So, $D_t \subseteq E^p(N \cup j, w; t, x') \subseteq D_t \cup \{S \cup j : S \in D_t\}$, and from this one easily shows that $E^p(N \cup j, w; t, x')$ is T-balanced if and only if D_t is T-balanced.

In the following we will show that $B_t = D_{dt}$ for $t > 0$, and $B_t = D_t$ for $t \leq 0$, so that these observations actually prove the theorem. Therefore, consider the following two cases.

- $t > 0$: Observe that B_t, and also D_{dt}, only contain multi-player coalitions. Let $S \in B_t$. Then $t \leq p_S^N(v(S) - x(S)) = (p_S^{N\cup j}/d)(v(S) - x(S))$, implying $S \in D_{dt}$. Therefore, $B_t \subseteq D_{dt}$.

  Now suppose $S \in D_{dt}$. If $S \in E^p(N \cup j, w; dt, x')$, then $dt \leq p_S^{N\cup j}(v(S) - x(S)) = dp_S^N(v(S) - x(S))$, implying $S \in B_t$, and if $S \cup j \in E^p(N \cup j, w; dt, x')$, then $dt \leq p_{S\cup j}^{N\cup j}(v(S \cup j) - x(S \cup j)) = p_{S\cup j}^{N\cup j}(v(S) - x(S)) = p_S^N(v(S) - x(S))$, again implying $S \in B_t$. We conclude that $B_t = D_{dt}$.

- $t \leq 0$: Let $S \in B_t$. Using (5), either $p_S^{N\cup j}(v(S) - x(S)) = p_S^N(v(S) - x(S)) \geq t$, or $p_{S\cup j}^{N\cup j}(v(S\cup j) - x(S\cup j)) = p_S^N(v(S) - x(S)) \geq t$, implying $S \in D_t$; i.e., $B_t \subseteq D_t$.

  Now suppose $S \in D_t$. If $v(S) - x(S) \geq 0$ then, evidently, $S \in B_t$. So, suppose further that $v(S) - x(S) < 0$. If $S \in E^p(N\cup j, w; t, x')$, then $t \leq p_S^{N\cup j}(v(S) - x(S)) \leq p_S^N(v(S) - x(S))$, and if $S \cup j \in E^p(N \cup j, w; t, x')$, then $t \leq p_{S\cup j}^{N\cup j}(v(S \cup j) - x(S \cup j)) \leq p_S^N(v(S) - x(S))$, and both because of (5). We conclude from this that $S \in B_t$, so that we must have $B_t = D_t$.

$\square$

The next theorem shows that weight systems corresponding to weighted nucleoli, having the NPO-property, are characterized by means of a multiplicative formula for the weights of the multi-player coalitions and a restrictive condition on the weights of one-player coalitions.

Theorem 5 *The weight system $p = (p_S^N)$ defines a weighted nucleolus with the NPO-property if and only if there exist constants $\gamma_j \in (0, 1]$, $j \in \mathcal{N}$, and $\alpha_N > 0$, $N \in \mathcal{F}$, such that*

$$p_S^N = \alpha_N \prod_{j\in S} \gamma_j \quad \text{for all } N \in \mathcal{F},\ S \subset N,\ |S| > 1, \tag{7}$$

and

$$p_{\{i\}}^N = \alpha_N \min(\gamma_i, \frac{p_{\{i\}}^{N\cup j}}{\gamma_j \alpha_{N\cup j}}) \quad \text{for all } N \in \mathcal{F},\ i \in N,\ j \notin N. \tag{8}$$

Proof:
The "if" case is obviously true. Thus, suppose ν^p has the NPO-property, i.e., $p = (p_S^N)$ fulfils (3) for each $N \in \mathcal{F}$, (4) and (5). Our first concern is the proof of (7). For $S \subset N$, $|S| > 1$, and $j \notin N$ we have

$$p_{S\cup j}^{N\cup j} = \frac{1}{c_{N,j}} p_S^N = \frac{1}{c_{N,j} d_{N,j}} p_S^{N\cup j}. \tag{9}$$

We consider two cases:

- $|N| > 3$: denote $1/(c_{N/k,k} d_{N/k,k})$ by $\beta_{N,k}$ for $k \in N$; then, applying (9) as many times as needed, we obtain

$$p_S^N = \frac{p_{\{i,j\}}^N}{\beta_{N,i}\beta_{N,j}} \prod_{k \in S} \beta_{N,k}$$

  for each $S \subset N$, and distinct players i and j of S. This shows that $p_{\{i,j\}}^N/(\beta_{N,i}\beta_{N,j})$ is not dependent on the choice of $i, j \in S$, and also it is not dependent on S because S does not occur in the expression; so, denote $p_{\{i,j\}}^N/(\beta_{N,i}\beta_{N,j})$ by α_N. Further,

$$d_{N,j} = \frac{p_S^{N \cup j}}{p_S^N} = \frac{\alpha_{N \cup j}}{\alpha_N} \prod_{k \in S} \frac{\beta_{N \cup j,k}}{\beta_{N,k}}, \tag{10}$$

  and this for arbitrary multi-player coalitions S. Now, applying (10) on both S and $S \cup k$, for $k \in N/S$, one easily obtains the expression $d_{N,j} = d_{N,j}\frac{\beta_{N \cup j,k}}{\beta_{N,k}}$. This shows that $\beta_{N \cup j,k}/\beta_{N,k}$ equals 1 for each $k \in N$, and with induction it is clear that $\beta_{N,k}$ is constant over the player sets N containing k, say $\gamma_k = \beta_{N,k}$. Observe that this constant is at most 1 since we must have non-increasing weights wrt. coalition inclusion. We conclude that (7) holds.

- $|N| = 3$: Take a player $j \notin N$ (this is possible since the cardinality of the universal player set $\mathcal{N}$ is larger than 3). Then, with $S = \{i, k\} \subset N$,

$$p_S^N = \frac{1}{d_{N,j}} p_S^{N \cup j} = \frac{\alpha_{N \cup j}}{d_{N,j}} \gamma_i \gamma_k.$$

  We have chosen $j \in \mathcal{N}/N$ arbitrarily, so that $\alpha_{N \cup j}/d_{N,j}$ must be constant over all players j outside N. Take α_N equal to this fraction, showing that also in this case the weights obey the form asserted in the theorem.

As for the one-player coalitions we know by now that $p_{\{i\}}^N$ equals the expression $c_{N,j} \min(p_{\{i,j\}}^{N \cup j}, p_{\{i\}}^{N \cup j})$, $c_{N,j}$ equals $1/(d_{N,j}\gamma_j)$, $d_{N,j} = \alpha_{N \cup j}/\alpha_N$, and $p_{\{i,j\}}^{N \cup j} = \alpha_{N \cup j}\gamma_i\gamma_j$. After substitution one obtains (8). □

Validity of the multiplicative formula in an indiscriminatory way, i.e.

$$p_S^N = \alpha_N \cdot \prod_{j \in S} \gamma_j \text{ for all } N \in \mathcal{F}, S \subset N,$$

suffices for (7) and (8) to hold. But as observed earlier, one-player coalitions have no substantial complaints about imputations. Hence NPO may obtain even if the weight system apparently discriminates against one-player coalitions. For instance, let $p_S^N = 1$ for all $N \in \mathcal{F}, S \subset N, |S| > 1$ and $p_{\{i\}}^N = 1/2$ for all $N \in \mathcal{F}, i \in N$. Then (7) and (8) are fulfilled with $\gamma_j = 1$ for all $j \in \mathcal{N}$ and $\alpha_N = 1$ for all $N \in \mathcal{F}$.

5 Conclusions and Ramifications

We have seen that the null player property for weighted nucleoli imposes mild, albeit significant restrictions on the weight system. In contrast, the NPO-property imposes severe restrictions on the weight system, but is not uniquely — up to proportionality — determining it. Weighted nucleoli are a possible response to the objection that the standard nucleolus is treating the excesses of any two coalitions as equally important for no apparent reasons. Yet theory alone does not provide a definitive answer to what constitutes an appropriate weight system. It has to be decided on *a priori* grounds, what constitutes, if any, a plausible weight system for each specific social choice problem. An axiomatization along the lines of Mascher et al. (1992) would not identify a weight system either. However, it distinguishes the (weighted) nucleolus as the unique solution concept, if the weight system is given, weighted excesses are accepted as measures of coalitional dissatisfaction, and three consistency properties across social choice problems: *non-discrimination*, *redundancy*, and *restriction to the least core* are postulated. The first two properties are quite intuitive and simple. The last property is more questionable and motivated by the desire to find a solution concept that points to the location of the core or to its latent image; see the discussions in Maschler et al. (1992) and Shubik (1983).

Like the standard nucleolus, weighted nucleoli share two properties:

(A) They minimize maximal excesses — roughly speaking.

(B) They serve as "core-pointers".

If one is willing to abandon (B), then one is no longer restrained to measure coalitional frustration or dissatisfaction in a linear way. An approach along this line may be found in Faigle et al. (1995). Maschler (1992) raises the possibility that on equity grounds, (A) might be given up as well and be replaced by tradeoff considerations across coalitions. The solution concepts proposed by Spinetto (1974), Ruiz et al. (1996), and Sudhölter (1997) can be viewed as incarnations of such "super-equity" ideas.

Given the multitude of solution concepts suggested in the literature, which one should one choose? Skeptics may conclude that cooperative game theory is inconclusive and leave it at that. Certainly, it would be presumptuous to claim that there is a single solution for all purposes. To quote Maschler (1992, p. 616):

> *"The real question, in my opinion, is not whether a particular solution is good or bad, but rather: In what circumstances should it be recommended and what insight would it then yield?"*

Investigations like ours hopefully help select a specific solution under specific circumstances.

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