Payoff functions in the one-way flow model of network formation for which Nash networks exist

Jean Derks and Martijn Tennekes*

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Abstract

We study a non-cooperative model of unilateral network formation . Derks et al. (2008b) prove the existence of local-Nash and global-Nash networks for games with payoff functions that satisfy a framework of axiomatic properties. In this paper we fully characterize one-way flow payoff functions that satisfy these properties. A one-way flow payoff function consists of profits from being connect to other agents and link costs.

Keywords: Non-cooperative Games, Network Formation, Axiomaization, Payoff functions.
JEL classification: C72, D85

1 Introduction

Network formation has become a major topic in game theoretic research in recent years. Agents or individuals (for instance human beings or companies) are represented as the nodes in a network, and a link between two agents represents some kind of relation between them, e.g. a friendship, cooperation or acquaintanceship. The formation of these networks can be modelled as follows. Agents, who are assumed to be autonomous, decide with whom they want to have a direct relation. Following some procedure where agents have the opportunity to form links among them, a network will be formed.

Pioneering work in this field has been done by Myerson (1977), Aumann and Myerson (1988), Myerson (1991), Jackson and Wolinsky (1996) and Bala and Goyal (2000a). For an overview of the literature of network formation we refer to Jackson (2005) and Van den Nouweland (2005). This paper follows the line of research introduced by Bala and Goyal (2000a) and continued by Bala and Goyal (2000b), Galeotti et al. (2006),Galeotti (2006),Haller and Sarangi (2005) and Haller et al. (2007).

*Corresponding author: martijn.tennekes@micc.unimaas.nl

In the model of Bala and Goyal (2000a) agents can form a link unilaterally, i.e., without consent of the other agents. A link between agents i and j that agent i forms is represented by an arc poining at i. On the basis of the formed network, each agent gets a payoff, which consists of a costs and a profits part. Each agent pays some costs for each link that he formed, i.e. for each link that is pointing at him directly in the final network. For the profits part two different settings are considered, the one-way and the two-way flow model. In the one-way flow model, agent i receives a certain profit from being connected to agent j, which is the case if and only if a directed path from j to i exists. In the two-way flow model agent i receives profits in the same way, but here paths may be undirected as well.

In this paper we focus on the one-way flow setting only. Bala and Goyal (2000a) introduced this model. They prove Nash existence of games where profits and link costs are homogeneous. Furthermore, they characterize the architecture of these Nash networks. The one-way flow model has been generalized by Galeotti (2006) by allowing profits and link costs heterogeneity. He characterizes strict-Nash networks for various settings of heterogeneity. The existence of Nash network for owner-homogeneous link costs, i.e. all links have equal costs with respect to the agent who formes them, has been proven by Derks et al. (2008a) and simultaneously by Billand et al. (2007). Furthermore, Derks et al. (2008a) provide a counterexample for a game with heterogeneous link costs for which Nash networks do no exist. In this example the link costs can be chosen arbitrarily close to the situation of owner-homogeneity.

A model of unilateral network formation which is inspired by the one-way flow model is studied by Derks et al. (2008b). They prove Nash existence for games with payoff fuctions that satisfy a specific set of axiomatic properties. Premisses for choosing those properties in that paper are that they are intuitive and that they define a class of payoff functions which is as large as possible in order to be able to prove the existence of Nash networks. Furthermore, they show that payoff functions with owner-homogeneous link costs and heterogeneous profits satisfy this set of properties. Derks et al. (2008b) use a local approach, where agents are restricted to play local actions, which are adding, deleting and replacing one link. They define local-Nash networks as networks in which each agent is playing a best local response. Local-Nash networks exist for games with payoff functions that satisfy a specific set of properties, and global-Nash exist for games with payoff functions that also satisfy another property.

In this paper we characterize the class of payoff functions with heterogeneous profits and hetereogeneous link costs that satisfy the set of properties, for which local-Nash and global-Nash networks exist.

2 Model

Let N be a finite set of agents. A link from agent j to i is denoted as (j, i). A *network* is defined as a set of links. Formally, a network is defined by $g \subseteq N \times N$ on the fixed set of agents N where loops are not allowed, i.e. $(i, i) \notin g$ for all

$i \in N$. Let $\mathcal{G}$ be the set of all these networks.

For a set union we will use the symbol '+', and for a set difference we will use the symbol '-'. Furthermore, these set operations are applied from left to right. For instance, the notation $g - g' + (j,i)$ equals $(g \setminus g') \cup \{(j,i)\}$.

Let $N_i(g)$ be the set of agents that i *observes* in g, i.e. the set of agents from whom a directed path to i exists in g and let $N_i^d(g)$ be the set of agents from whom a link pointing at i exists in g.

We define an *action* of agent i as a set of agents, denoted as $S_i \subseteq N \setminus \{i\}$. The network, after i chooses to link with the agents in S_i, is described by

$$g_{-i} + \{(j,i) : j \in S_i\}.$$

Here, $g_{-i} = g \setminus g_i$ is the network g with all i's links removed. The union of all the actions of all agents in N define the outcome network.

We distinguish between global and local responses. A global action is equivalent to the action that we defined above, i.e. a global action of agent i to network g is a replacement of i's links in g by a new set of links. A local action of agent i to network g is one of these three elementarily adjustments: the addition, deletion or replacement of a link.

A global action S_i of agent i is called a *best global response* if

$$\pi_i\big(g_{-i} + \{(j,i) : j \in S_i\}\big) \geq \pi_i\big(g_{-i} + \{(j,i) : j \in T_i\}\big),$$

for all global actions T_i. A network g is called a *global-Nash network* if $N_i^d(g)$ is a best global response for all $i \in N$. Analog definitions apply for the local case.

Let $\mathrm{Car}(g)$, the carrier of g be the set of agents who are begin- and/or endpoints of a link in g. A *component* h of network g is defined as a network $h \subseteq g$ where an undirected path exists between any two agents $i, j \in \mathrm{Car}(h)$ in network h and there does not exist a link in g with one begin- or endpoint in $\mathrm{Car}(h)$ and one begin- or endpoint outside $\mathrm{Car}(h)$. Let g^j denote the (unique) component of g where j is contained. Let $g_{-ij} = g^j_{-i} + (j,i)$, thus the component in g_{-i} where j is contained, with link (j,i) added.

Let v_{ij} be the *profits* that i receives from being connected to j, which is the case if and only if a direct path exists from j to i. We shall use the following shorthand notation: $v_i(S) = \sum_{j \in S} v_{ij}$. The *costs* a link (j,i) are c_{ij}. Let $c_i(S) = \sum_{j \in S} c_{ij}$. Agent i's *payoff* is denoted as $\pi_i : \mathcal{G} \to \mathbb{R}$. We consider the following class of payoff functions, proposed by Galeotti (2006), where the profits and the link costs are heterogeneous:

$$\pi_i(g) = v_i(N_i(g)) - c_i(N_i^d(g)) \tag{1}$$

We refer to payoff functions in this class as B&G-functions. The intrinsic value v_{ii} for each agent i is often non-zero in literature. However, we may apply the transformation $\pi_i'(g) = \pi_i(g) - \pi_i(g_{-i}) = \pi_i(g) - v_{ii}$ without loss of generality. Therefore we may assume that $v_{ii} = 0$.

Bala and Goyal (2000a), Galeotti (2006) and Derks et al. (2008a) assume that the c and v values are non-negative or strictly positive. However, in this paper we drop these restrictions to incorporate all payoff functions of the form (1).

Derks et al. (2008a) and Billand et al. (2007) simultaneously proved the existence of global-Nash networks for B&G functions with owner-homogeneous link costs, i.e. $c_{ij} = c_i$ for all i, j. Derks et al. (2008b) continued this line of research by introducing a framework of indepedent axiomatic payoff properties.

Before describing the properties proposed by Derks et al. (2008b), we need the following definitions. A network or component is called *proper* if every agent has at most one outgoing link. An agent is called a *topagent* if he observes all agents in his component. Finally, a link (j, i) is called *beneficial* if $\pi_i(g_{-ij}) \geq \pi_i(g_{-i})$.

This framework consists of the following properties:

Property DA Payoff function π satisfies **DA**(*disjoint additivity*), if for each two networks g and g', disjoint w.r.t. an agent i (that is, $\text{Car}(g) \cap \text{Car}(g') \supseteq \{i\}$), we have

$$\pi_i(g \cup g') = \pi_i(g) + \pi_i(g').$$

Property NA We say that π satisfies **NA**(*naturality*) if $\pi_i(g + (k, i)) \leq \pi_i(g)$ whenever there is a directed path from k to i in the network g.

Property BT Payoff function π satisfies **BT**(*beneficial topagent*) if the following holds. Let link (k, i) be beneficial, and suppose there are topagents in the component g^k_{-i}. Then there is a topagent j in g^k_{-i} such that $\pi_i(g_{-ij}) \geq \pi_i(g_{-ik})$.

Property BF Payoff function π satisfies **BF**(*beneficial furthest*) if the following holds. Let link (k, i) be beneficial, let the component g^k_{-i} be proper and let agent i be active in g^k_{-i} (thus $i \in \text{Car}(g_{-i})$, which implies that there is a link in g^k_{-i} with beginpoint i). Then link (j, i) is beneficial for agent j furthest away from i, i.e., the longest directed path starting at i ends at j.

Property BG Payoff function π satisfies **BG**(*beneficial growth*) if $\pi_i((g+(k,r))_{-ij}) \geq \pi_i((g+(k,r))_{-i})$ for each pair of agents $k,\ r$, whenever $\pi_i(g_{-ij}) \geq \pi_i(g_{-i})$.

Property BS Payoff function π satisfies **BS**(*beneficial shrink*) if $\pi_i((g-(k,r))_{-ij}) \geq \pi_i((g-(k,r))_{-i})$ whenever $\pi_i(g_{-ij}) \geq \pi_i(g_{-ij})$ and link (k, r) is a spoke in g, in other words, when a directed cycle containing k and r is present in $g - (k, r)$.

We call a payoff function *ordinary* if it satisfies all these properties. Derks et al. (2008b) prove that local-Nash networks always exist for games with ordinary payoff functions. Furthermore, they show that these properties are independent and that B&G-functions with homogeneous link costs are ordinary. In this paper we characterize the class of ordinary B&G-functions. Here, profits and link costs are heterogeneous and are not restricted to be strictly positive as in Bala and Goyal (2000a) and Galeotti (2006), but they may be negative as well.

3 Characterization

We provide necessary and sufficient conditions for B&G functions for being ordinary. Notice that a link is beneficial if and only if $c_{ij} \leq v_i(S)$, where S is the set of agents that i observes by using (j, i).

First of all, it is clear that all B&G-functions satisfy **DA** and **BS**.

It is also clear that the property **NA** is satisfied if and only if $c_{ij} \geq 0$ for all $i, j \in N$. We identify the non-negativity of the link costs by the property NNC:

NNC (short for *non-negative costs*) $\quad c_{ij} \geq 0 \quad$ for all $i, j \in N$.

Property **BG** is satisfied if $v_{ij} \geq 0$ for all $i, j \in N$. Indeed, if $g' = g + (k, r)$, then the set of agents that agent i observes in $(g')_{-ij}$ is a superset of the set of agents that i observes in g_{-ij}. It follows that if all profits are non-negative, then (i, j) is beneficial in g' whenever it is beneficial in g. The follwing example shows that the reverse does not hold, i.e. **BG** does not imply the non-negativity of the profits.

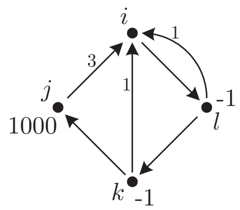

Figure 1: An example where some v values are negative but where **BG** is still satisfied.

Example 1 *Consider network g as depicted in Figure 1. The numbers near the nodes are the profits for agent i: $v_{ij} = 1000$ and $v_{ik} = v_{il} = -1$. The numbers near the links are the link costs: $c_{ij} = 1$ and $c_{ik} = c_{il} = 3$. In all networks on the agent set $\{i, j, k, l\}$, link (j, i) is beneficial. Links (k, i) and (l, i) are only beneficial in networks where j is observed by k and l respectively. In these networks, the beneficiality of (k, i) respectively (l, i) is guaranteed when adding other links. Hence, we conclude that **BG** is satisfied while some profits are negative.*

The following theorem provides a full characterization of property **BG** with respect to B&G payoff functions.

Theorem 2 *Let π be defined as in Equation 1. Then π satisfies **BG** if and only if the following property holds:*

PBG If $c_{ij} \leq v_i(S)$ for agent set $S \subset N$ and agent $j \in S$, then $c_{ij} \leq v_i(S')$ for all $S' \supset S$.

Proof. Suppose that PBG does not hold. Then there exists a set $S \subset N$ and a set $S' \supset S$ such that $c_{ij} \leq v_i(S)$ and $c_{ij} > v_i(S') = v_i(S) + v_i(T)$ for some $i \in S$ with $T = S' \setminus S$. Consider network g where the agents in S form a component with j as topagent and the agents in T form a component with k as topagent. The beneficiality of (j,i) is $\pi_i(g_{-ij}) = v_i(S) - c_{ij} \geq 0$. Now consider the network $g + (k,j)$. Link (j,i) is not beneficial in this network, because $\pi_i((g+(k,j))_{-ij}) = v_i(S) + v_i(T) - c_{ij}$ which is strictly less than zero, since we assumed that $c_{ij} > v_i(S')$. Hence **BG** is not satisfied.

Now suppose that **BG** is not satisfied. Then a network g exists which does not contain link (k,r) where $\pi_i(g_{-ij}) \geq 0$ and $\pi_i((g+(k,r))_{-ij}) < 0$. Let S be the set of agents that i observes in g_{-ij} and S' be the set of agents that i observes in network $(g+(k,r))_{-ij}$. Clearly $S' \supseteq S$. Since $\pi_i(g_{-ij}) \neq \pi_i((g+(k,r))_{-ij})$, it follows that $S' \neq S$ and hence that $S' \supset S$. Since $\pi_i(g_{-ij}) = v_i(S) - c_{ij} \geq 0$ and $\pi_i((g+(k,r))_{-ij}) = v_i(S') - c_{ij} < 0$, it follows that PBG does not hold. □

For the remaining properties **BT** and **BF** we have the following results.

Theorem 3 *Let π be defined as in Equation 1. Then π satisfies **BT** if and only if the following property holds:*

PBT If $c_{ij} \leq v_i(S)$ for agent set $S \subseteq N$ and agent $j \in S$, then $c_{ij} \geq c_{ik} - v_i(T)$ for all sets $T \subseteq N \setminus S$ and $k \in T$.

Proof. Suppose that PBT does not hold. Then agents j and k and agent sets $S \subseteq N$ and $T \subseteq N \setminus S$ exist where $j \in S$ and $k \in T$ such that $c_{ij} \leq v_i(S)$ and $c_{ij} < c_{ik} - v_i(T)$. Consider the network depicted in Figure 2. Here j observes each agent in S. All agents in T form a path with k being the head of it. Since $c_{ij} \leq v_i(S)$, link (j,i) is beneficial in this network, i.e., $\pi_i(g_{-ij}) = v(S) - c_{ij} \geq 0$. Furthermore, we have

$$\begin{aligned}\pi_i(g_{-ij}) &= v(S) - c_{ij} \\ &> v(S) - c_{ik} + v_i(T) \\ &= v(S \cup T) - c_{ik} \\ &= \pi_i(g_{-ik}).\end{aligned}$$

Since k is the only topagent in the component g^j_{-i}, we have shown that **BT** is not satisfied.

Now suppose that **BT** is not satisfied. Then a network g exists where (j,i) is beneficial and where a topagent exist in g^j_{-i} such that for each topagent k in g^j_{-i} we have $\pi_i(g_{-ik}) < \pi_i(g_{-ij})$. Let $S = N_i(g_{-ij})$ be the set of agents that i observes via (j,i) and let T be the set of all other agents in the component g^j_{-i}. Take a topagent k; it follows that $k \in T$ and that $S \cup T = N_i(g_{-ik})$. We have

$$\begin{aligned}0 &> \pi_i(g_{-ik}) - \pi_i(g_{-ij}) \\ &= v_i(S \cup T) - c_{ik} - (v_i(S) - c_{ij}) \\ &= c_{ij} - (c_{ik} - v_i(T)).\end{aligned}$$

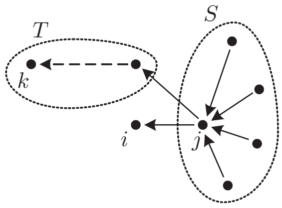

Figure 2: Network for which PBT does not hold.

Hence we have that $c_{ij} < c_{ik} - v_i(T)$. Since (j, i) is beneficial in g it follows that $c_{ij} \leq v_i(S)$. Therefore PBT does not hold. This completes the proof. $\square$

Theorem 4 *Let π be defined as in Equation 1. Then π satisfies **BF** if and only if the following property holds:*

PBF If $c_{ij} \leq v_i(S)$ for agent set $S \subseteq N$ and agent $j \in S$, then $c_{ik} \leq v_i(S')$ for all $S' \supseteq S$, $k \in S' \setminus j$

Proof. Suppose that PBF does not hold. Then sets S and $S' \supseteq S$ and agents $j \in S$ and $k \in S' - j$ exist where $c_{ij} \leq v_i(S)$ and where $c_{ik} > v_i(S')$. Consider a network g where agent i observes all agents in S in g_{-ij} and where i observes all agents in S' in g_{-ik}. We may assume that k is the agent who is furthest away from i. Link (j, i) is beneficial because $c_{ij} \leq v_i(S)$ and link (k, i) is not beneficial because $c_{ik} > v_i(S')$. Since we assumed that k is the furthest away from i it follows that **BF** is not satisfied.

Now suppose that **BF** is not satisfied. Then a network g exists that contains a proper component g^j_{-i} where i is active, link (j, i) is beneficial, k is the furthest away from i and where link (k, i) is not beneficial. Let S' be the set of agents in g^j_{-i} and let $S \subseteq S'$ be the set of agents that j observes in this component. Since (j, i) is beneficial it follows that $c_{ij} \leq v_i(S)$. Agent k is a topagent, because this component is proper. Therefore $\pi_i(g_{-ik}) = v_i(S') - c_{ik}$. We assumed that (k, i) is not beneficial. Therefore $c_{ik} > v_i(S')$. Hence the property PBF does not hold. $\square$

The properties PBG and PBF exhibit similarities; both properties tell that if the costs for a link of i are less or equal than $v_i(S)$ for some $S \subseteq N$, then according to PBG, the costs for the same link are less or equal than $v_i(S')$, for all $S' \supset S$, and according to PBF, the costs for any other link from an agent in S' to i are less or equal than $v_i(S')$ for all $S' \subset S$. In the following theorem we show that PBF implies PBG.

Theorem 5 *In the class of B&G-functions, the property PBG is implied by PBF.*

Proof. Let π by a B&G-function that satisfies PBF. Let i, j be agents and let $S \subseteq N$ where $j \in S$. Let S' be a strict superset of S and let k be an agent in $S' \setminus S$. Assume that $c_{ij} \leq v_i(S)$. Then, according to PBF we have that $c_{ik} \leq v_i(S')$. According to PBF again, this latter inequality implies that $c_{ij} \leq v_i(S')$. By our assumption that $c_{ij} \leq v_i(S)$, we conclude that property PBG holds. □

To summarize our results thus far, we know that properties **DA** and **BS** are satisfied by all B&G functions, property **NA** requires that link costs are non-negative (NNC), and that PBF implies PBG. Hence we know that properties NNC, PBT and PBF are sufficient to characterize the class of ordinary B&G functions. Now we show that these properties are also necessary for this characterization.

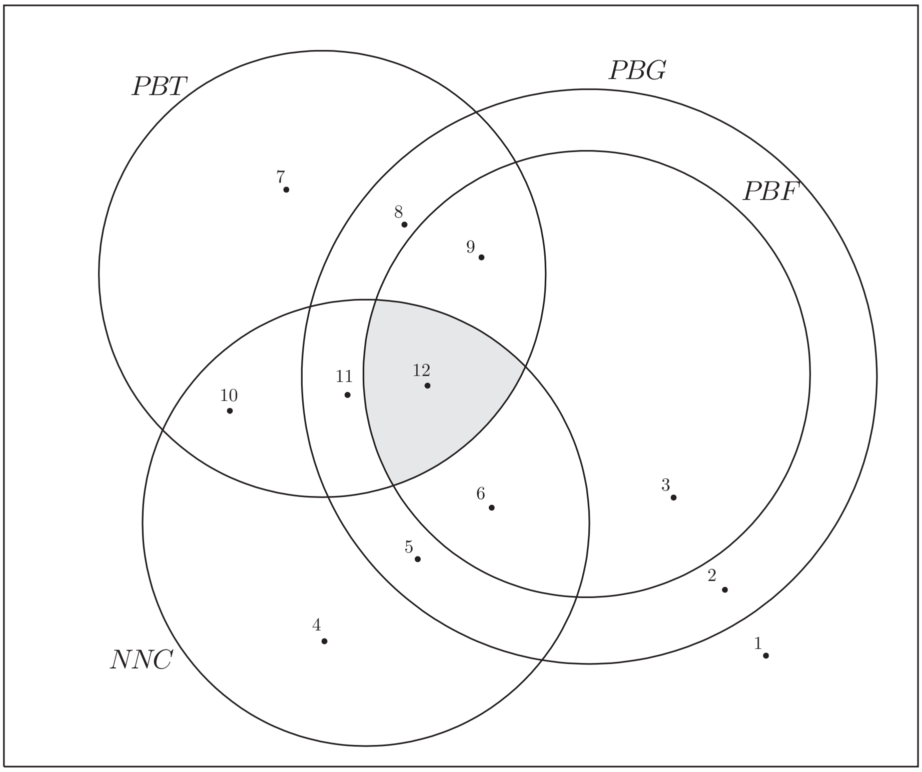

Figure 3: The characterization of B&G functions. The grey area corresponds to the ordinary B&G functions.

In Figure 3 we show the characterization of B&G-functions graphically. Here, we draw four cycles, that correspond to the properties PBT, NNC, PBF and PBG. The intersections of these cycles create twelve area's. These area's are labelled from 1 to 12. Vertices in a cycle correspond to B&G-functions which satisfy the corresponding property and vertices outside the cycle correspond to B&G-functions which do not satisfy the corresponding property. For instance, all vertices in the area 7 are contained in the cycle of PBT but they are outside all other three cycles. Thus, the payoff functions in this area satisfy PBT, but not the other three properties.

In Table 1, we describe for each area which properties are satisfied and we provide an example that proofs nonemptyness of that area. All these examples are three-person games. For area's 9 and 12 we use two examples each, to which we will refer in the next section.

The grey area (number 12) in Figure 3 is the area that corresponds to B&G-functions that are orderly, i.e., that satisfy all properties in our framework. For an example of a orderly B&G-function, we used a trivial one (12a) with all c and v values equal to 0. Example 12b shows that B&G-functions with hetrogeneity among agents and link costs can be orderly as well.

Since we found examples for area's 6, 9 and 11, we have shown that the properties NNC, PBT and PBF are sufficient and necessary for the characterization of orderly B&G-functions. Thus, we have the following corollary:

Corollary 6 *Let π be a B&G payoff function. Then π is orderly if and only if*

$$\text{(NNC)} \qquad c_{ij} \geq 0 \qquad\qquad \forall j \in N, \text{ and}$$

Table 1: Characterization of the twelve area's in Figure 3

Area	Properties				Example			
	PBT	NNC	PBF	PBG	c_{ij}	c_{ik}	v_{ij}	v_{ik}
1	n	n	n	n	-1	0	0	-2
2	n	n	n	y	-1	1	0	-1
3	n	n	y	y	-1	3	2	1
4	n	y	n	n	0	2	0	-1
5	n	y	n	y	0	2	0	1
6	n	y	y	y	1	3	2	1
7	y	n	n	n	1	-2	3	-3
8	y	n	n	y	-1	3	-2	2
9a	y	n	y	y	-1	-1	3	3
9b	y	n	y	y	1	-1	3	1
10	y	y	n	n	3	0	4	-2
11	y	y	n	y	2	3	1	1
12a	y	y	y	y	0	0	0	0
12b	y	y	y	y	1	2	4	1

if $c_{ij} \leq v_i(S)$ *for agent set* $S \subseteq N$ *and agent* $j \in S$ *then*

$$\begin{array}{lll} \textit{(PBT)} & c_{ij} \geq c_{ik} - v_i(T) & \forall T \subseteq N \setminus S, k \in T, \textit{ and} \\ \textit{(PBF)} & c_{ik} \leq v_i(S') & \forall S' \supseteq S, k \in S' - j \end{array}$$

By Derks et al. (2008b) we know that local-Nash networks always exist when payoff functions are orderly. Therefore we have the following corollary.

Corollary 7 *Let* π *be a B&G-payoff function that satisfies NNC, PBT and PBF. Then local-Nash networks always exist.*

The following example shows that the converse is not true.

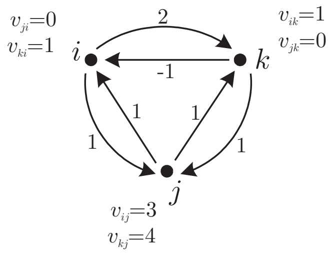

Figure 4: Example.

Example 8 *Let* π *be a B&G-function where the profits and link costs are defined as in Figure 4. Here, the numbers next to the links are the link costs. Notice that agent i's payoff is equal to the example of area 9 (see Table 1). Thus,* π_i *does not satisfy NNC, but does satisfy PBT and PBF. It can be easily checked that* π_j *and* π_k *satisfy PBT and PBF and NNC. The network* $g = \{(k, i), (j, k)\}$*, depicted in Figure 5, is (local-)Nash.*

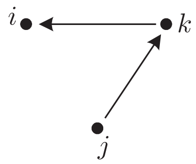

Figure 5: Nash network.

4 Downstream efficiency

Derks et al. (2008b) show that for orderly payoff functions that also satisfy an additional property, glocal-Nash networks exist. This additional property is called downstream efficiency (**DE**) and is defined as

Property DE Payoff function π satisfies **DE** (short for *downstream efficiency*) if

$$\pi_i(g + (k,i)) \leq \pi_i(g + (j,i))$$

for any network g where $(j,i) \notin g$ and $(k,i) \notin g$ and where a directed path exists from k to j in g_{-i}.

The following theorem shows that this property implies homogeneous link costs and non-negative profits.

Theorem 9 *Let π be defined as in Equation 1. Then π satisfies **DE** if and only if*

OHC(short for owner-homogeneous costs)	$c_{ij} = c_{ik}$	$\forall j,k \in N$;
NNP(short for non-negative profits)	$v_{ij} \geq 0$	$\forall j,k \in N$.

Proof. First we proof that **DE** implies OHC. Assume that **DE** is satisfied and consider a network g, where all agents in $N \setminus \{i\}$ are contained in a directed cycle and i is isolated. Let j and k be agents in $N \setminus \{j\}$. Since there exists a directed path in both ways, we have by **DE** that $\pi_i(g + (j,i)) = \pi_i(g + (k,i))$. Therefore it follows that $c_{ij} = c_{ik}$.

We show that **DE** also implies NNP. Suppose that **DE** is satisfied. Let j be an agent in $N \setminus \{i\}$. Consider a network g where links (k,j) and (k,i) exist and where there is no path from j to k. Then by **DE** we have

$$0 \leq \pi_i(g + (j,i) - (k,i)) - \pi_i(g) = c_{ik} - c_{ij} + v_{ij}.$$

By OHC it follows that $v_{ij} \geq 0$.

Now we proof that OHC and NNP imply **DE**. Suppose that **DE** is not satisfied. Then a network g exists where $\pi_i(g + (j,i)) < \pi_i(g + (k,i))$, where (j,i) and (k,i) do not exist and where a directed path from k to j exists in g_{-i}. We may assume there is no path vice versa since otherwise this inequality implies that OHC is violated. Let $S \subset N$ be the set of agents that i observes in network $g + (j,i)$ and not in $g + (k,i)$. Then we have

$$0 > \pi_i(g + (j,i)) - \pi_i(g + (k,i)) = v_i(S) + c_{ij} - c_{ik}$$

If OHC is satisfied, then it follows that $v_i(S) < 0$. Hence an agent l exists in S such that $v_{ij} < 0$. This contradicts NNP. □

Theorem 10 *The properties OHC and NNP imply PBT and PBF.*

Proof. Let π be a payoff function which satisfies OHC and NNP.

By NNP we have that $v_i(T) \geq 0$ for all $T \subseteq N$. Furthermore, by OHC, $c_{ij} = c_{ij}$ for all $j,k \in N$. Therefore if follows that $c_{ij} \geq c_{ik} - v_i(T)$ and hence that PBT is satisfied.

Let $S \subseteq N$ be a set of agents and $j \in S$ be an agent such that $c_{ij} \leq v_i(S)$. By NNP it follows that $c_{ij} \leq v_i(S')$ for all $S' \supseteq S$. By OHC it follows that $c_{ik} = c_{ij} \leq v_i(S')$, for all $k \in N$ and therefore that PBF is satisfied. □

Examples 9a, 9b, 12a and 12b show that neither NNC implies OHC and NNP nor vice versa. Figure 6 provides a graphical characterization of OHC and NNP. By Derks et al. (2008b) we know that the play of the dynamic game always converges to a global-Nash network when payoff functions are orderly and satisfy **DE**. Therefore we have the following corollary.

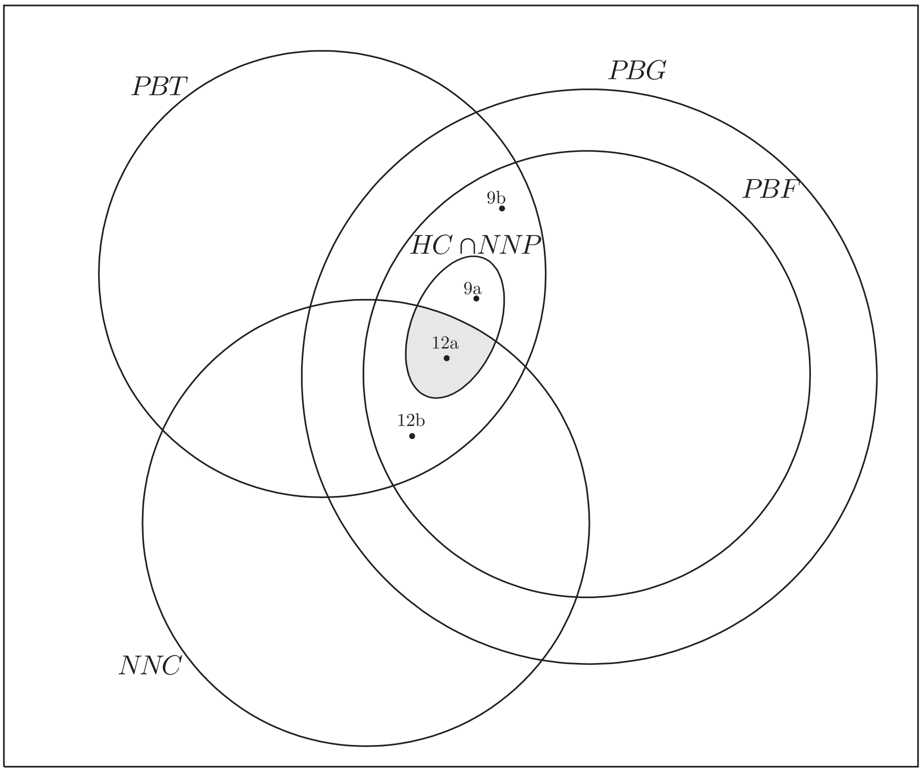

Figure 6: The characterization of B&G functions. The grey area corresponds to the B&G functions that satisfy OHC, NNP and NNC.

Corollary 11 *Let* π *be a B&G-payoff function. If it satisfies*

$$\begin{array}{llll} \textit{(NNC)} & c_{ij} \geq & 0 & \forall j \in N, \textit{ and} \\ \textit{(OHC)} & c_{ij} = & c_{ik} & \forall j,k \in N, \textit{ and} \\ \textit{(NNP} & v_{ij} \geq & 0 & \forall j \in N, \end{array}$$

then global-Nash networks exist, and furthermore the procedure defined by A-1 to A-3-c always converges.

Thus, the global-Nash existence results and the convergence result provided by Derks et al. (2008b) only apply to B&G functions where link costs are owner-homogeneous and both link costs and profits are non-negative.

References

Aumann, R., Myerson, R., 1988. Endogenous formation of links between players and coalitions. In: Roth, A. (Ed.), The Shapley Value. Cambridge University Press.

Bala, V., Goyal, S., 2000a. A non-cooperative model of network formation. Econometrica 68, 1181–1229.

Bala, V., Goyal, S., 2000b. A strategic analysis of network reliability. Review of Economic Design 5, 205–228.

Billand, P., Bravard, C., Sarangi, S., 2007. Existence of nash networks in one-way flow models, forthcoming in Economic Theory.

Derks, J., Kuipers, J., Tennekes, M., Thuijsman, F., 2008a. Existence of one-way flow Nash networks, forthcoming in Neogy S. K. et al. (eds.), Modeling, Computation and Optimization, World Scientific.

Derks, J., Kuipers, J., Tennekes, M., Thuijsman, F., 2008b. Local dynamics in network formation. Tech. rep., Maastricht University, Department of Mathematics.

Galeotti, A., 2006. One-way flow networks: the role of heterogeneity. Economic Theory 29 (1), 163–179.

Galeotti, A., Goyal, S., Kamphorst, J., 2006. Network formation with heterogeneous players. Games and Economic Behavior 54 (2), 353–372.

Haller, H., Kamphorst, J., Sarangi, S., 2007. (Non-)existence and scope of Nash networks. Economic Theory 31 (3), 597–604.

Haller, H., Sarangi, S., 2005. Nash networks with heterogeneous links. Mathematical Social Sciences 50 (2), 181–201.

Jackson, M. O., 2005. A survey of the formation of networks: stability and efficiency. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Ch. 1.

Jackson, M. O., Wolinsky, A., 1996. A strategic model of social and economic networks. Journal of Economic Theory 71 (1), 44–74.

Myerson, R., 1977. Graphs and cooperation in games. Math. Operations Research 2, 225–229.

Myerson, R., 1991. Game Theory: Analysis of Conflict. Harvard University Press, Cambridge, Massachusetts.

Van den Nouweland, A., 2005. Models of network formation in cooperative games. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Ch. 2.