

Null Players Out?

Linear Values for Games with Variable Supports ^{*†}

J.J.M. Derks [‡] and H.H. Haller [§]

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Abstract

The paper studies the consequences of the Null Player Out (NPO) property for single-valued solutions on the class of cooperative games in characteristic function form. We allow for variable player populations (supports or carriers). A solution satisfies the NPO property, if elimination of a null player does not affect the payoffs of the other players. Our main emphasis lies on individual values. For linear values satisfying the null player property and a weak symmetry property, necessary and sufficient conditions for the NPO property are derived.

*Key words: transferable utility games, Null Players Out, linear values, individual values, weak symmetry.

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[‡]Jean Derks, Department of Mathematics, Faculty of General Sciences, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands (mailing address)

[§]Hans Haller, Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0316, USA

1 Introduction

Here we deal with the standard notion of a cooperative game in characteristic function form, but consider variable player sets (carriers or supports). The characteristic function value represents the maximum aggregate utility the members of each possible coalition of players can achieve through cooperation among themselves. Thus any conceivable negative or positive externalities coalition members might impose on each other, have been netted out already. Moreover, a characteristic function conventionally describes a game with side-payments and perfectly transferable utility. That means that utility can freely and without loss be transferred across coalition members. These are strong and, at times, implausible assumptions. But when they are satisfied, one can proceed with thorough analysis. Then “null players”, that is individuals whose net contribution to any coalition is zero, can be defined in a precise way. Attributes like “dummy” or “null” suggest irrelevance. In a specific sense, the net effect of social interaction with players of that kind is zero, indeed. For that reason, dummies are often treated as redundant in a solution or index: a dummy ends up simply with the utility she can achieve by herself. More specifically, the “null player axiom” or “null player property” allocates zero utility to null players in a utility allocation or assigns zero power to null players in a power index. *A priori* the null player axiom treats null players differently from other players. But combined with other axioms, this property often implies that each player receives or is judged according to some weighted average of his net contributions.

While the null player property means that a null player neither gains nor benefits in excess of his (zero) net contribution, the power distribution or utility allocation within the rest of society can still be affected by the presence of a null player. This has been illustrated in Haller (1994) with respect to collusive agreements. Removing a null player may affect the power indices of other players. To see why this is so, consider a game, a coalition of players

in the game, a select player of the game not belonging to the coalition, and a probabilistic value for the select player in this game. The select player makes a certain contribution to the given coalition. Next consider the game that is obtained by adding a null player to the original player set and consider a probabilistic value for the select player in the enlarged game. Now the select player makes the previous contribution twice: to the previously given coalition and to the previous coalition joined by the novice null player. As a rule then, the individual value calculated for the select player is sensitive to the introduction of the null player, unless the probabilities in the enlarged game and the probabilities in the original game are related in the very specific way of equation (3) below.

Since in general, the presence of a null player can have effects on other players, it makes sense to investigate the impact of the “null player out” property, abbreviated NPO in the sequel, which requires that a null player does not influence a power distribution or utility allocation within the rest of society. The focus of this paper lies on the implications of the NPO property for linear values. A companion paper, Derks and Haller (1998) is entirely devoted to the study of weighted nucleoli with the null player property and NPO.

Under the label “reduction axiom”, the NPO-property plays a crucial rôle in the comparison of two types of collusion in Haller (1994). Tijs and Driessen (1986) derive a “Dummy Players Out” property for the τ -value. Also, for linear values, the NPO-property is an implication of population monotonicity. Furthermore, a “carrier-free” or “support-free” definition of games and values as in Dubey, Neyman, and Weber (1981) presumes the NPO property.

First and foremost, our curiosity about the NPO-property has been inspired by comparisons of two types of collusion between two players: fusion

or amalgamation à la Lehrer (1988) and a proxy or representation agreement à la Haller (1994). In a proxy agreement, both players continue to exist physically; one of them becomes a null player, whereas the proxy player gets the power to act and sign on behalf of both of them. Amalgamation means that the number of players is reduced; the two colluding players are replaced by a single player who acts and signs on behalf of both of them. Technically, the case of amalgamation differs from the proxy case in that in the former type of collusion the designated null player is removed. Haller (1994) demonstrates that this seemingly minor difference can be crucial. More precisely, he considers (in the standard sense) symmetric probabilistic group values and the effect of collusion on the sum of the values of the two colluding players. He finds that the difference between proxy agreement and amalgamation does not matter if, and only if, a reduction axiom equivalent to the NPO-property is satisfied.

The outline of the paper is now as follows. Section 2 reviews the underlying concepts. Especially, we will elaborate on the notions of individual versus group values. For a more comprehensive introduction we refer to Weber (1988). Section 3 is devoted to the NPO property of individual values. Our central result is a characterization of the NPO property for values that are linear and satisfy the null player and a weak symmetry property. Such a value has the NPO property if, and only if, it can be reconstructed from the weight(s) the player attaches to make it on his own, without seeking cooperation. Section 4 extends our findings to group values. In some respects, these results resemble those of Dubey, Neyman, and Weber (1981) who provide a representation of a “semivalue” by means of the moments of a probability distribution on the unit interval. We relate and compare the two approaches. Finally, section 5 offers some conclusions.

2 Preliminaries

In the first place, we consider finite non-empty sets of players and the space of cooperative games in characteristic function form for each set of players. A *value* or, more precisely, an *individual value* is a real valued function on the game space. Frequently, a value expresses the worth of a game to one of the players involved. In specific contexts, a value may be interpreted differently. It may measure the value added by the player or constitute an index of the power of the player. The main focus of our formal investigation is on such individual values. This has been done to some extent in Weber (1988) whose fundamental research provides an important stepping stone for our own inquiry.

A *group value* is defined as a collection of values, one value for each player involved. A group value corresponds to the standard notion of a single-valued solution concept. In contrast with the usual approach we do not necessarily assume any a priori interconnection among the individual values such as symmetry and efficiency.

We consider a non-empty universal player set \mathcal{N} with generic elements i, j, \dots . Let \mathcal{F} denote the set of finite non-empty subsets N of \mathcal{N} . To each $N \in \mathcal{F}$, we associate \mathcal{G}^N , the set of cooperative transferable utility games with player set N . Every $v \in \mathcal{G}^N$ is identified with its characteristic function, i.e. $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. Hence $\mathcal{G}^N \equiv \{v \in \mathbb{R}^{2^N} : v(\emptyset) = 0\}$. Then $\mathcal{G} \equiv \bigcup_{N \in \mathcal{F}} \mathcal{G}^N$ is the game space. For $N \in \mathcal{F}$, $v \in \mathcal{G}^N$, and $M \subset N$, we allow that v may be considered as a game in \mathcal{G}^M .

For $i \in \mathcal{N}$, define $\mathcal{F}_i = \{N \in \mathcal{F} : i \in N\}$. An (*individual*) *value* for $i \in \mathcal{N}$ is a mapping $\psi_i : \bigcup_{N \in \mathcal{F}_i} \mathcal{G}^N \rightarrow \mathbb{R}$, i.e. a value is a real valued function on the set of games in which the player participates. Let Ψ_i denote the set of individual values for i , with generic elements ϕ_i, ψ_i, \dots . Let $\Psi = (\Psi_i)_{i \in \mathcal{N}}$ denote the set of (*universal*) *group values*, with generic elements ϕ, ψ, \dots .

We adopt the following notational conventions. For $j \in \mathcal{N}$, $S \in \mathcal{F}$, $S \setminus j$

is shorthand for $S \setminus \{j\}$. \subseteq denotes weak set inclusion, whereas \subset denotes strict (proper) set inclusion.

Often, it is convenient to consider local individual values ϕ_i or (local) group values $\phi = (\phi_i)_{i \in N}$ with restricted domain \mathcal{G}^N for some $N \in \mathcal{F}$. We shall do so when appropriate. The notation $\phi(N, \cdot)$ is used, if we want to emphasize a local group value with domain \mathcal{G}^N .

Next we will examine several concepts for (individual) values in more detail.

2.1 Monotonicity

Let us fix momentarily a player set N . We say that a game v is *i-monotonic* for a player $i \in N$, if the worth of a coalition increases when player i joins the coalition, i.e.,

$$v(T \cup \{i\}) \geq v(T) \quad \text{for each } T \subseteq N \setminus \{i\}.$$

A value ϕ_i for player i is *individually monotonic*, if $\phi_i(v) \geq 0$ for each i -monotonic game v . Player i is called a *null player* in the game v if both v and $-v$ are i -monotonic, i.e., a null player does not influence the worth of a coalition he enters. A value ϕ_i is said to have the *null player property* if $\phi_i(v) = 0$ whenever i is a null player in v ; it is implied by the individual monotonicity property plus oddness. (ϕ_i is odd, if $\phi_i(-v) = -\phi_i(v)$ for all v .)

A dummy player in a game v is closely related to the null-player notion. Player i is a *dummy* if he is a null-player in $v - v(\{i\})u_i$, where u_i is the *i-dictator game*, i.e., $u_i(S) = 1$ if coalition S contains player i , otherwise $u_i(S) = 0$. Observe that the null-player property is implied by the *dummy player property*: $\phi_i(v) = v(\{i\})$ whenever i is a dummy in v .

2.2 Linearity

Consider a *linear* value ϕ_i , i.e. for $N \in \mathcal{F}_i, v \in \mathcal{G}^N$:

$$\phi_i(N, v) = \phi_i(N, \sum_{T \subseteq N} v(T) \mathbf{1}^T) = \sum_{T \subseteq N} v(T) \phi_i(N, \mathbf{1}^T) = \sum_{T \subseteq N} a_{N,T}^i v(T),$$

where $a_{N,T}^i$ stands for the value of player i in the *unity game* $(N, \mathbf{1}^T)$, i.e., $\mathbf{1}^T(S) = 1$ for the coalition $S = T$, otherwise $\mathbf{1}^T(S) = 0$. The most prominent of the before-mentioned properties can be translated into properties on the weight system $a_N^i = (a_{N,T}^i)_{T \subseteq N}$:

- the null-player property is equivalent to:
 $a_{N,T}^i = -a_{N, T \cup \{i\}}^i$ for each $T \subseteq N \setminus \{i\}$;
- individual monotonicity is equivalent to: $a_{N,T}^i \geq 0$ for each non-empty $T \subseteq N$.

Sometimes, we allow for the possibility that all values are normalized in the sense that for (each player set and) each player there is one game on which all values attain the same numerical value. The dictator games are well suited for this purpose. From now on, whenever we opt for such a *normalization*, we shall assume that

$$\phi_i(u_i) = 1$$

holds for each value ϕ_i .

A linear value satisfies:

- the normalization assumption is equivalent to: $\sum_{T \subseteq N \setminus \{i\}} a_{N, T \cup \{i\}}^i = 1$;
- the dummy player property is equivalent to the combination of the null player property and the normalization assumption.

Well known linear values are the probabilistic values. A value ϕ_i for player i is a *probabilistic value* if there exist probability weights (i.e., non-negative and summing up to 1) $p_{N,T}^i, T \subseteq N \setminus \{i\}$, such that

$$\phi_i(N, v) = \sum_{T \subseteq N \setminus \{i\}} p_{N,T}^i \cdot [v(T \cup i) - v(T)].$$

A group value is probabilistic, if it is composed of probabilistic values. Using the above translations of value properties into properties of the weight system makes it evident that

Theorem 1 (Weber 1988) *ϕ_i is a probabilistic value if, and only if, ϕ_i is linear, individually monotonic, and fulfils the dummy player property.*

2.3 Symmetry

If π is a permutation of the player set N and v is a game on N , then $v \circ \pi$ corresponds to the game where player j takes on the rôle of player $\pi(j)$ in v . A value ϕ_i for player i is called *individually symmetric*, if it is independent of the names of the other players, i.e., $\phi_i(N, v) = \phi_i(N, v \circ \pi)$ for each permutation π of N such that $\pi(i) = i$.

A group value $\phi = (\phi_i)_{i \in N}$ is *symmetric* in the usual sense, if it satisfies $\phi_{\pi(i)}(N, \pi v) = \phi_i(N, v)$ for all permutations π of N , where $\pi v(S) \equiv v(\pi^{-1}S)$. A symmetric group value reflects the situation that each player values a game in the same way as the others. Symmetry for a group value therefore implies that the components are individually symmetric. Of course, a group value composed of individually symmetric values need not be symmetric.

Within the class of probabilistic group values, Weber (1988; Th. 10) yields the following characterization of symmetric values.

Theorem 2 *A probabilistic group value ϕ is symmetric if, and only if, the corresponding weight system p fulfils $p_{N,T}^i = p_{N,S}^j$ for any two players i and j and coalitions T and S with the same cardinality.*

A value ϕ_i is called *weakly symmetric*, if its value of $\mathbf{1}^{\{i\}} \in \mathcal{G}^N$ only depends on the number $n = |N|$ of players involved. Weak symmetry for each of its components falls short of symmetry of a group value. In the sequel we will combine weak symmetry with *basic symmetry*, requiring for $\phi = (\phi_i)_{i \in N}$ that each player i values the unity game $\mathbf{1}^{\{i\}}$ equally, i.e., $\phi_i(N, \mathbf{1}^{\{i\}}) = \phi_j(N, \mathbf{1}^{\{j\}})$ for all $i, j \in N$. As we will see lateron, the combination of weak symmetry and basic symmetry implies symmetry under certain circumstances.

3 The NPO Property for Individual Values

We are now especially interested in values ϕ with the property

$$\phi_i(N, v) = \phi_i(N \setminus \{j\}, v) \quad (1)$$

for all $i, j \in N$ and $v \in \mathcal{G}^N$ such that j is a null player in v and $i \neq j$. We call this the “*null player out*” property or **NPO**(-property) for short.

We aim at characterizing the individual values that satisfying linearity, the null player property, weak symmetry, and the NPO property.

Let us commence by examining the NPO-property in combination with the null player property and linearity. The latter two properties for a value ϕ_i imply that this value can be written as

$$\phi_i(N, v) = \sum_{T \subseteq N \setminus \{i\}} p_{N, T}^i(v(T \cup \{i\}) - v(T)). \quad (2)$$

If we further impose the NPO-property we obtain

$$\begin{aligned} & \sum_{T \subseteq N \setminus \{i, j\}} p_{N \setminus \{j\}, T}^i(v(T \cup \{i\}) - v(T)) = \sum_{T \subseteq N \setminus \{i\}} p_{N, T}^i(v(T \cup \{i\}) - v(T)) \\ = & \sum_{T \subseteq N \setminus \{i, j\}} [p_{N, T}^i(v(T \cup \{i\}) - v(T)) + p_{N, T \cup \{j\}}^i(v(T \cup \{i, j\}) - v(T \cup \{j\}))] \\ = & \sum_{T \subseteq N \setminus \{i, j\}} (p_{N, T}^i + p_{N, T \cup \{j\}}^i)(v(T \cup \{i\}) - v(T)) \end{aligned}$$

for each game v with null player j , and especially for the games (N, w_T) , $T \subseteq N \setminus \{i, j\}$, defined by $w_T(S) = 1$ for $S = T \cup \{i\}$ and $S = T \cup \{i, j\}$, and $w_T(S) = 0$ otherwise. Therefore, we obtain

$$p_{N \setminus \{j\}, T}^i = p_{N, T}^i + p_{N, T \cup \{j\}}^i \quad \text{for each } N \in \mathcal{N} \text{ and } T \subseteq N \setminus \{i, j\}. \quad (3)$$

In case (3) holds, one easily proves the NPO-property. We conclude that

Theorem 3 *Suppose a value ϕ_i satisfies linearity and the null player property. Then (2) holds. Moreover, ϕ_i satisfies the NPO property if, and only if, (3) holds.*

Adding weak symmetry to linearity, the null player property, and NPO, has a strong implication: individual symmetry results. Assuming weak symmetry in addition to linearity, the null player property, and NPO, we may put

$$\gamma_n^i = p_{N, \emptyset}^i = \phi_i(N, \mathbf{1}^{\{i\}}),$$

i.e., γ_n^i is the weight player i attaches to making it on his own, without seeking cooperation in a local population N of size n . Now (3) implies

$$\gamma_{n-1}^i = \gamma_n^i + p_{N, \{j\}}^i$$

or

$$p_{N, \{j\}}^i = \gamma_{n-1}^i - \gamma_n^i \quad (4)$$

for all players i and $j \neq i$ in N . Using again (3), for any 1-person coalition $T = \{k\}$, together with (4) yields

$$\gamma_{n-2}^i - \gamma_{n-1}^i = \gamma_{n-1}^i - \gamma_n^i + p_{N, \{j, k\}}^i$$

or

$$p_{N, \{j, k\}}^i = \gamma_{n-2}^i - 2\gamma_{n-1}^i + \gamma_n^i. \quad (5)$$

Continuing in this way all weights can be described using $\gamma_n^i, \dots, \gamma_1^i$:

Lemma 1 *Suppose a value ϕ_i satisfies (2) and (3). If ϕ_i is weakly symmetric, denoting $\phi_i(N, \mathbf{1}^{\{i\}}) = \gamma_{|N|}^i$, then*

$$p_{N,T}^i = \sum_{k=0}^t \binom{t}{k} \gamma_{n-k}^i (-1)^{t-k} \quad (6)$$

for all $i \in N$, $T \subseteq N \setminus \{i\}$, $n = |N|$, $t = |T|$.

Proof. The assertion is shown by induction to the number of players in N and T . For $n + t = |N| + |T| \leq 5$ we have proved it already. Now suppose (6) holds for all player sets N and coalitions T such that $|N| + |T| < r$. Let N be a player set and T a coalition with $|N| + |T| = r$. We may assume that $n = |N| > 2$ and $n > t = |T| \geq 1$. Let $i \in N \setminus T$, $j \in T$. Then

$$\begin{aligned} p_{N,S}^i &= p_{N \setminus \{j\}, T \setminus \{j\}}^i - p_{N, T \setminus \{j\}}^i \\ &= \sum_{k=0}^{t-1} \binom{t-1}{k} \gamma_{n-1-k}^i (-1)^{t-1-k} - \sum_{k=0}^s \binom{t-1}{k} \gamma_{n-k}^i (-1)^{t-1-k} \\ &= \sum_{k=1}^t \binom{s-1}{k-1} \gamma_{n-k}^i (-1)^{t-k} + \sum_{k=0}^s \binom{t-1}{k} \gamma_{n-k}^i (-1)^{t-k} \\ &= \sum_{k=1}^t \left(\binom{s-1}{k-1} + \binom{s-1}{k} \right) \gamma_{n-k}^i (-1)^{t-k} + \gamma_n^i (-1)^t \\ &= \sum_{k=1}^t \binom{s}{k} \gamma_{n-k}^i (-1)^{t-k} + \gamma_n^i (-1)^t \end{aligned}$$

This proves the general assertion. \square

An immediate consequence is

Corollary 1 *Suppose a value ϕ_i satisfies (2) and (3). If ϕ_i is weakly symmetric, then it is individually symmetric and, moreover, the weights in (2) depend only on population size $n = |N|$ and coalition size $t = |T|$ and are obtained via an explicit formula on the weights of the empty set, $\gamma_n^i = p_{N, \emptyset}^i$.*

Going through the proofs of Lemmas 3 and 1 in reverse, one obtains converse statements. Hence we have

Theorem 4 *Suppose ϕ_i satisfies linearity and the null player property. Then the following are equivalent:*

- (i) *NPO and weak symmetry hold.*
- (ii) *There exist scalars $\gamma_1^i, \gamma_2^i, \dots$ such that the weight system p in (2) satisfies (6).*

In case that (i) and (ii) do hold, $\phi_i(N, \mathbf{1}^{\{i\}}) = \gamma_{|N|}^i$.

Notice that so far we have not assumed a probabilistic value, i.e.

- (a) the weights $p_{N,T}^i$ are nonnegative, and
- (b) $\sum_{T \subseteq N \setminus \{i\}} p_{N,T}^i = 1$ for each $N \in \mathcal{F}$.

First of all, a (in the usual sense) symmetric probabilistic group value ϕ is determined by weights $p_{n,t}, 0 \leq t \leq n-1$, with $\sum_{t=0}^{n-1} \binom{n-1}{t} p_{n,t} = 1$ for all $n \in N$. By Lemma 1, our results encompass the crucial finding of Haller (1994; Proposition 9): ϕ has the **NPO**-property if, and only if, the **weight equation** “ $p_{n-1,t} = p_{n,t} + p_{n,t+1}$ ” holds for $n \geq 2, n-2 \geq t \geq 0$.

Secondly, for $n \in \mathbb{N}$, let $\mathbb{N}(n) = \{m \in \mathbb{N} : m \geq n\}$. Define $X = \bigcup_{n \in \mathbb{N}} \mathbb{R}^{IN(n)}$ and the difference operator $\Delta : X \rightarrow X$ by

$$\Delta x = ((\Delta x)_{n+1}, (\Delta x)_{n+2}, \dots) = (x_n - x_{n+1}, x_{n+1} - x_{n+2}, \dots) \in \mathbb{R}^{IN(n+1)}$$

for $x = (x_n, x_{n+1}, \dots) \in \mathbb{R}^{IN(n)}$. Solving equation (3) for $p_{N, T \cup \{j\}}^i$ conveys that the array $\gamma^i = (\gamma_1^i, \gamma_2^i, \dots)$ generates arrays

$$\begin{aligned} \gamma^{(0)} &= \gamma^i \in \mathbb{R}^{IN(1)}, \\ \gamma^{(1)} &= \Delta \gamma^{(0)} \in \mathbb{R}^{IN(2)}, \\ \gamma^{(2)} &= \Delta \gamma^{(1)} \in \mathbb{R}^{IN(3)}, \end{aligned}$$

etc. which determine the weights as follows:

$$p_{N,T}^i = h_{n,t}^i = \gamma_{|N|}^{(t)}. \quad (7)$$

Condition (a) thus implies that all these arrays are nonnegative, i.e., γ^i is nonnegative and each of the derived arrays $\gamma^{(t)}, t \geq 0$, is decreasing. Let us call the array $\gamma^i = (\gamma_n^i)_{n \in \mathbb{N}}$ *totally decreasing* if all these conditions are met.

Thirdly, (6) implies

$$\sum_{T \subseteq N \setminus \{i\}} p_{N,T}^i = \sum_{t=0}^{n-1} \sum_{k=0}^t \binom{n-1}{t} \binom{t}{k} \gamma_{n-k}^i (-1)^{t-k} = \sum_{k=0}^{n-1} \delta_{n-1,k} \cdot \gamma_{n-k}^i, \quad (8)$$

$$\text{with } \delta_{m,k} = \sum_{t=k}^m \binom{m}{t} \binom{t}{k} (-1)^{t-k}.$$

Lemma 2 $\delta_{m,m} = 1$ and $\delta_{m,k} = 0$ for $k \neq m$.

Proof. The amount $\binom{m}{t} \binom{t}{k}$ equals $\sum_{T \subseteq M, |T|=t} \sum_{K \subseteq T, |K|=k} 1$, with M a set of m elements; therefore,

$$\begin{aligned} \delta_{m,k} &= \sum_{t=k}^m \sum_{T \subseteq M, |T|=t} \sum_{K \subseteq T, |K|=k} (-1)^{t-k} \\ &= \sum_{T \subseteq M, |T| \geq k} \sum_{K \subseteq T, |K|=k} (-1)^{|T \setminus K|} = \sum_{K \subseteq M, |K|=k} \sum_{R \subseteq M \setminus K} (-1)^{|R|}. \end{aligned}$$

Now

$$\sum_{R \subseteq S} (-1)^{|R|} = \begin{cases} 1, & \text{if } |S| = 0; \\ 0, & \text{if } |S| = 1. \end{cases}$$

The assertion follows. \square

Given (8), according to Lemma 2 validity of (b) amounts to

$$\gamma_1^i = 1. \quad (9)$$

Now we can summarize our main insight as follows.

Theorem 5 *Let ϕ_i be an individual value. Then the following are equivalent.*

- (i) *ϕ_i is probabilistic and weakly symmetric and has the NPO property.*
- (ii) *ϕ_i is probabilistic and individually symmetric and has the NPO property.*
- (iii) *There exists a totally decreasing array γ^i satisfying (9) such that ϕ_i is given by (2) and (7).*

Notice that in particular, an individual value with the properties listed in the theorem is probabilistic. Therefore it is linear and satisfies the dummy player axiom. Hence the normalization $\phi_i(u_i) = 1$ has to hold. Since the probability vectors representing probabilistic values form a simplex of dimension $2^{n-1} - 1$, there remain many degrees of freedom for additional normalizations. Setting $\phi_i(N, \mathbf{1}^{\{i\}}) = \gamma_{|N|}^i$ is but one of them. Thus $\gamma_{|N|}^i$ stands for the worth to a player of participating in his idiosyncratic unanimity game. By forming the coalition $\{i\}$, player i can reap the payoff $\mathbf{1}^{\{i\}}(\{i\}) = 1$ all for himself. $\gamma_{|N|}^i = 1$ for all N means that player i is, indeed, completely self-centered and out to capture the payoff of 1, while foregoing possible gains from cooperation at other occasions. At the other extreme, a player with $\gamma_1^i = 1, \gamma_2^i = \gamma_3^i = \dots = 0$ appears totally unselfish, resting content with zero payoff in his idiosyncratic unanimity games with at least one other player. More generally, $\gamma_{|N|}^i$ can be interpreted as the weight player i attaches to making it on his own, without seeking cooperation in a local population N of size n .

A few instructive examples follow. Taking the corresponding totally decreasing arrays as the starting points, additional totally decreasing arrays can be obtained by the operations of shifting, first differencing, adding, and rescaling.

Example (MINIMUM VALUE):

Let $p_{N,T}^i = 1$, if $T = \emptyset$; $p_{N,T}^i = 0$, if otherwise.

Then $\gamma_n^i = 1$ for all n and $\phi_i(N, v) = v(\{i\})$ for all (N, v) .

As discussed before, such a player attaches maximal weight on making it on his own. This attitude serves him very well in his idiosyncratic unanimity games $\mathbf{1}^{\{i\}}$. But he does not fare that well in many games where cooperation would be rewarding.

Example (UTOPIA VALUE):

Let $p_{N,T}^i = 1$, if $T = N \setminus i$; $p_{N,T}^i = 0$, if otherwise.

Then $\gamma_1^i = 1$, $\gamma_n^i = 0$ for $n \geq 2$, and $\phi_i(N, v) = v(N) - v(N \setminus i)$ for all (N, v) .

Referring to our earlier discussion, such a player attaches minimum weight on making it on his own as long as there is another player present in the game. This attitude serves him badly in his idiosyncratic unanimity game $\mathbf{1}^{\{i\}}$. He fares better than a completely self-centered type of player in many games where cooperation can be rewarding. However, his reward may often be modest, because of his unconditional eagerness to cooperate.

Example (SHAPLEY VALUE): Cf. Haller (1994), Ex. 5:

The weights $p_{n,t}^i = \frac{1}{n} \cdot \binom{n-1}{t}^{-1}$ yield the Shapley value. The associated array is $\gamma^i = \left(\frac{1}{n}\right)_{n \in IN}$. Such a player is more eager to cooperate or share with the other players as the size of the local player population increases.

Example (BANZHAF VALUE): Cf. Haller (1994), Ex.6:

Fix $\beta \in (0, 1)$ and set $p_{n,t}^i = \beta^{n-1-t} \cdot (1-\beta)^t$. Then $\gamma_n^i = \beta^{n-1}$.

If $\beta = 1/2$, then ϕ^i is the Banzhaf value:

$p_{n,t}^i = (1/2)^{n-1}$ and $\gamma_n^i = (1/2)^{n-1}$.

As the size of the local player population increases, this player becomes more eager to cooperate. The propensity to cooperate grows faster than in the case of the Shapley value.

4 The NPO Property for Group Values

Consider a local or global group value $\phi = (\phi_i)$. Even if each ϕ_i shares all the properties listed in Theorem 5, it can still be the case that

- (i) ϕ is not symmetric and
- (ii) different ϕ_i are generated by different arrays.

But all it takes to tie the individual values closely together is the additional requirement of basic symmetry: $\phi_i(N, \mathbf{1}^{\{i\}}) = \phi_j(N, \mathbf{1}^{\{j\}})$ for all i, j . It is straightforward to show

Theorem 6 *Let ϕ_i be a group value. Then the following are equivalent.*

- (i) *ϕ is probabilistic and weakly symmetric and satisfies basic symmetry and the NPO property.*
- (ii) *ϕ_i is probabilistic and symmetric and has the NPO property.*
- (iii) *There exists a totally decreasing array γ satisfying (9) such that for each player i , ϕ_i is given by (2) and (7), with $\gamma^i = \gamma$.*

As mentioned in the introduction, the NPO property is implicitly assumed in a “carrier-free” or “support-free” definition of games and values as in Dubey, Neyman, and Weber (1981). They define a game by means of a characteristic function $V : 2^{\mathcal{N}} \rightarrow \mathbb{R}$ with $V(\emptyset) = 0$. A set $S \subseteq \mathcal{N}$ is a *support* or *carrier* of V , if for each $T \subseteq \mathcal{N}$, $V(T) = V(S \cap T)$. Let G denote the vector space of all games with finite support and let, for $N \in \mathcal{F}$, G^N denote the subspace of G consisting of the games with support N . Let AG denote the subspace of additive games in G and AG^N denote the subspace of additive games in G^N . AG^N is isomorphic to \mathbb{R}^N and identified with \mathbb{R}^N , whenever convenient.

Given a bijection Π of \mathcal{N} onto itself, define the game Π_*V by $\Pi_*V(S) = V(\Pi S)$. Call V *monotonic*, if $S \subseteq T$ implies $V(S) \leq V(T)$.

A *semivalue* on G is a mapping $\varphi : G \rightarrow AG$ such that

1. φ is linear;
2. $\varphi\Pi_* = \Pi_*\varphi$ for each bijection of \mathcal{N} onto itself;
3. if V is monotonic, then φV is monotonic;
4. if $V \in AG$, then $\varphi V = V$.

What about the NPO property?

¶ First, observe that for any $N \in \mathcal{F}$, there is a canonical isomorphism $\sigma^N : \mathcal{G}^N \rightarrow G^N$ given by $\sigma^N v(S) = v(S \cap N)$ for $S \subseteq \mathcal{N}$.

¶ Secondly, $G^M \subseteq G^N$ for any $M, N \in \mathcal{F}$ with $M \subseteq N$.

¶ Thirdly, for any semivalue φ on G and any $N \in \mathcal{F}$, let φ^N denote the restriction of φ to G^N which then defines a local value $\phi(N, \cdot)$ on \mathcal{G}^N via

$$\phi_i(N, v) \equiv \varphi_i^N(\sigma^N(v)). \quad (10)$$

¶ Fourthly, each φ^N and consequently each local value $\phi(N, \cdot)$ inherits the local counterparts of the properties 1. — 4. from the semivalue φ . Hence each of these local values is linear, monotonic, symmetric and has the dummy player property. Therefore, they are all symmetric probabilistic values.

¶ Fifthly, consider $i, j \in \mathcal{N}, i \neq j, N \in \mathcal{F}_i \cap \mathcal{F}_j$, and $v \in \mathcal{G}^N$ with j as a null player. Then (1) holds, i.e. ϕ has the NPO property. The argument is that $(N \setminus \{j\}, v)$ is shorthand for the game $(N \setminus \{j\}, w)$ defined by

$w(T) = v(T)$ for $T \subseteq N \setminus \{j\}$ which satisfies

$$\begin{aligned} & \sigma^{N \setminus \{j\}} w(S) \\ &= w(S \cap (N \setminus \{j\})) = v(S \cap (N \setminus \{j\})) = v((S \cap N) \setminus \{j\}) = v(S \cap N) \\ &= \sigma^N v(S). \end{aligned}$$

Thus $\sigma^{N \setminus \{j\}} w = \sigma^N v$ and the assertion follows from (10).

¶ Finally, consider a symmetric probabilistic group value ϕ on \mathcal{G} that has the NPO property. Then the set of equations (10) can be used to define a semivalue φ on G . Therefore, we have established

Theorem 7 *Let ϕ be a symmetric probabilistic group value on \mathcal{G} . Then ϕ has the NPO property if, and only if, it is induced by a (uniquely determined) semivalue φ on G , that is a set of equations (10) holds.*

In view of Theorem 7, the concept of a semivalue implicitly assumes the NPO property and it is not surprising that an analogue of the weight equation (3) appears in the analysis of Dubey, Neyman, and Weber (1981). Further, the analysis of Dubey et al. parallels ours in another important aspect. They introduce arrays $\alpha = (\alpha_0, \alpha_1, \dots)$ by setting $\alpha_n = p_n^{n+1}$ for $n = 0, 1, \dots$. Like our γ 's, the α 's allow to recover all the local probability weights. The α 's are particularly well suited to derive a representation of a semivalue — or rather of the associated probability weights — by means of the moments of a probability distribution ξ on the unit interval. In fact, each array $\alpha = (\alpha_0, \alpha_1, \dots)$ qualifying as the determinant of a semivalue consists of the sequence of moments for a unique ξ . In contrast, the γ 's seem less suited for such a statistical interpretation. However, a term $1 - \gamma_n$ readily suggests itself for the game-theoretical interpretation as “propensity to cooperate” whereas we cannot think of a similar interpretation of the α 's.

5 Conclusions

At the center of our investigation are the repercussions of assuming that the presence of a null player does not affect the outcome for the other players. Our main emphasis lies on individual values. Specifically within the class of linear values with the null player property, we have characterized the values satisfying weak symmetry as the ones which can be recovered from special arrays of numbers $\gamma = (\gamma_n)$ via (6) or (7). In order to generate a probabilistic value, these arrays have to be totally decreasing and must fulfil (9). The elements of such an array reflect, in a sense, the weight a player attaches to making it on his own, without seeking cooperation in local populations of various sizes.

Our analysis shows that the NPO property in combination with other standard properties has non-trivial implications. As pointed out in 2.3, the NPO property follows from population monotonicity and oddness, hence in particular from population monotonicity and linearity. In Section 4, we argue that a carrier-free definition of a semivalue implicitly assumes the NPO property.

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