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Theory and Methodology

On linear inequality systems without strongly redundant constraints

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Abstract

For an $(m \times n)$-matrix A the set $C(A)$ is studied containing the constraint vectors b of $\mathbb{R}^m$ without strongly redundant inequalities in the system $(Ax \leq b,\ x \geq 0)$. $C(A)$ is a polyhedral cone containing as a subset the cone $Col(A)$ generated by the column vectors of A. This paper characterizes the matrices A for which the equality $C(A) = Col(A)$ holds. Furthermore, the matrices A are characterized for which $C(A)$ is generated by those constraint vectors $\beta^i \in C(A)$, $i \in \{1, 2, \ldots, m\}$, for which the feasible region $\{x \in \mathbb{R}^n_+ : Ax \leq \beta^i\}$ equals $\{x \in \mathbb{R}^n_+ : (Ax)_i \leq 1\}$. Necessary conditions are formulated for a constraint vector to be an element of $C(A)$. The class of matrices is characterized for which these conditions are also sufficient.

Keywords: Linear inequality systems; Strong redundancy; Polyhedral cones; Extreme directions

1. Introduction

In the theory of linear programming one is particularly interested in computational aspects. One of these aspects is the size reduction of the problem by considering redundancy. A constraint in a linear inequality system $(Ax \leq b,\ x \geq 0)$, with A an $(m \times n)$-matrix and b an m-vector, is *redundant* if the removal of this constraint from the system does not affect the feasible region $\{x \in \mathbb{R}^n_+ : Ax \leq b\}$. A constraint inequality, say $(Ax)_i \leq b_i$, is *strongly redundant* if it is redundant and for each feasible solution x we have strict inequality $(Ax)_i < b_i$. A comprehensive enumeration of methods for testing and determining redundancy can be found in Karwan e.a. [1] and Telgen [3].

Here, we will deal with linear inequality systems in which strongly redundant constraints are in fact absent.

A constraint vector b is called *sharp* if none of the constraints in $(Ax \leq b,\ x \geq 0)$ is strongly redundant. Thus, b is sharp if the feasible solution set $\{x \in \mathbb{R}^n_+ : Ax \leq b$ and $(Ax)_i = b_i\}$ (the *facet* corresponding to the i-th constraint) is non-empty for each $i \in \{1, 2, \ldots, m\}$.

Let $C(A)$ denote the set of all sharp constraint vectors with respect to the matrix A. To our knowledge this set has not been discussed before, and in this paper our primary concern is to obtain insight in the structure of the set $C(A)$. It turns out to be a polyhedral cone in $\mathbb{R}^m$ and, therefore, a finite generating set for $C(A)$ exists (cf. Klee [2]). Unfortunately, there are no direct methods available to construct such subsets in general. It is, however, possible to describe some of the sharp constraint vectors which are contained in any generating set, up to a scalar multiplication. These so-called extreme directions will be described in Section 3 where a sufficient condition is given for a column vector of A to

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be an extreme direction of $C(A)$. Also, the sharp constraint vector β^i for which $\{x \in \mathbb{R}^n_+ : Ax \le \beta^i\}$ equals $\{x \in \mathbb{R}^n_+ : (Ax)_i \le 1\}$, if existing, turns out to be extreme in $C(A)$, with $i \in \{1, 2, \dots, m\}$.

For two sharp constraint vectors b and c for the matrix A the algebraic sum of the corresponding feasible regions not necessarily corresponds to the vector $b + c$. For example, consider the (2×3)-matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix},$$

$b = (1, 2)^t$ and $c = (2, 1)^t$. The reader is invited to check that the sum $\{x \in \mathbb{R}^3_+ : Ax \le b\} + \{x \in \mathbb{R}^3_+ : Ax \le c\}$ equals the intersection $\{x \in \mathbb{R}^3_+ : Ax \le b + c\} \cap \{x \in \mathbb{R}^3 : (1, 1, 1)x \le 2\}$ which is strictly contained in $\{x \in \mathbb{R}^3_+ : Ax \le b + c\}$.

However, each facet of $\{x \in \mathbb{R}^n_+ : Ax \le b + c\}$ has a non-empty intersection with $\{x \in \mathbb{R}^n_+ : Ax \le b\} + \{x \in \mathbb{R}^n_+ : Ax \le c\}$. This follows from the observation that for $x' \in \{x \in \mathbb{R}^n_+ : Ax \le b,\ (Ax)_i = b_i\}$ and for $x'' \in \{x \in \mathbb{R}^n_+ : Ax \le c,\ (Ax)_i = c_i\}$, with $1 \le i \le m$ arbitrary, we have $x' + x'' \ge 0$, $A(x' + x'') \le b + c$ and $(A(x' + x''))_i = b_i + c_i = (b + c)_i$.

This observation shows that a decomposition of a sharp constraint vector b into a weighted sum of sharp constraint vectors from a well described generating set W, say $b' = \Sigma_{b \in W} p_b b$, can be of help in case the feasible regions of the decomposition parts b, $b \in W$, are known and one is interested in (some of) the elements in the facets of $\{x \in \mathbb{R}^n_+ : Ax \le b'\}$. In general this method is impractible since there exists no feasible decomposition method for arbitrary cases. In Sections 4, 5 and 6 three classes of matrices are characterized with a computationally clear structure, so that for these matrices decomposition methods exist and, thus, to which the above approach can be applied. The paper therefore provides a mathematical tool for a further examination of the above mentioned observations.

In Section 4 a necessary and sufficient condition is given in order to have equality between $C(A)$ and the cone $Col(A) = \{Ax : x \in \mathbb{R}^n_+\}$ which is generally a subset of $C(A)$.

Section 5 concerns the characterization of positive matrices A for which $C(A)$ equals the cone generated by the constraint vectors β^i, $i \in \{1, 2, \dots, m\}$. This characterization is considered in more detail for the case $n = 2$.

In Section 6 we will formulate an upper bound set for the sharp constraint set. For this upper bound set it is computationally easy to decide whether or not a constraint vector is an element of it. We will characterize the class of matrices for which the upper bound set coincides with the sharp constraint set.

In conclusion a summary is given of the results.

Throughout the paper we assume that A is an $(m \times n)$-matrix with coefficients a_{ik}, $i \in \{1, 2, \dots, m\}$, $k \in (1, 2, \dots, n\}$. x^t and A^t denote the transpose of a vector x and matrix A. The notations for the origin and a unity vector are 0 and e^i; the dimension of these vectors will be clear from the context.

2. The sharp constraint set

Let I^i denote the $(m \times m - 1)$-matrix obtained from the identity matrix by deleting the i-th column.

Lemma 1. *The sharp constraint set $C(A)$ equals the intersection of the cones generated by the column vectors of the $(m \times n + m - 1)$-matrices (A, I^i); $1 \le i \le m$.*

Proof. A constraint vector b is an element of $C(A)$ if and only if

$$\{x \in \mathbb{R}^n_+ : Ax \le b \text{ and } (Ax)_i = b_i\} \ne \emptyset \text{ for each } i \in \{1, 2, \dots, m\}. \tag{1}$$

Using slack variables (1) is equivalent to

$$b \in \{(A, I^i)z : z \in \mathbb{R}^{n+m-1}_+\} \quad \text{for each } i \in \{1, 2, \dots, m\}.$$

The lemma now follows. □

The sharp constraint set $C(A)$ is therefore a finite intersection of polyhedral cones $Col(A, I^i)$, $i \in \{1, 2, \ldots, m\}$, yielding

Theorem 2. *The sharp constraint set is a polyhedral cone.*

The dimension of $C(A)$ can easily be detected with the help of the following

Theorem 3. *$C(A)$ is fully dimensional if and only if no two rows of A are equal up to a scalar multiplication factor.*

Proof. Suppose that row i and row j are multiples of each other, say $(e^i)^t A = \alpha (e^j)^t A$, with $\alpha \in \mathbb{R}$. Let $y = \alpha e^j - e^i$ if $\alpha \geq 0$, otherwise we let y to be $e^i - \alpha e^j$. Note that $y^t z \geq 0$ for all elements z of $Col(A, I^i)$ and $y^t z \leq 0$ for all elements z of $Col(A, I^j)$. This implies $y^t z = 0$ for all $z \in \bigcap_{1 \leq k \leq m} Col(A, I^k) = C(A)$. Therefore, $C(A)$ is not fully dimensional.

On the other hand, suppose $C(A)$ is not fully dimensional. Consider the following two cases:

1) $Col(A, I^i)$ is not fully dimensional for a particular $i \in \{1, 2, \ldots, m\}$. This only occurs if row i of A is the zero vector, which is a multiple of each of the other rows.
2) The cones $Col(A, I^i)$, $i \in \{1, 2, \ldots, m\}$, are fully dimensional. Because its intersection, $C(A)$, is not fully dimensional, indices $i, j \in \{1, 2, \ldots, m\}$ exist such that $Col(A, I^i) \cap Col(A, I^j)$ is not fully dimensional. Let $y \in \mathbb{R}^m$ be such that $y^t z = 0$ for all $z \in Col(A, I^i) \cap Col(A, I^j)$. It follows that $y_k = 0$ for $k \in \{1, 2, \ldots, m\} \setminus \{i, j\}$. From $y^t A = 0^t$ it follows that row i and row j are multiples of each other. □

The following lemma characterizes those column vectors of A which are redundant in the determination of $C(A)$ in the sense that deleting these columns does not affect the sharp constraint set.

Lemma 4. *Let A' be the $(m \times n - 1)$-matrix obtained from A by deleting a column vector. Then $C(A') \subseteq C(A)$. Furthermore, $C(A') = C(A)$ if and only if the deleted column vector is a sharp constraint vector of A'.*

Proof. Note that $Col(A', I^i)$ is contained in $Col(A, I^i)$ for each $i \in \{1, 2, \ldots, m\}$. This shows the first part of the lemma.

To prove the second part, suppose that Ae^k is the deleted column vector. If $Ae^k \notin C(A')$ then $C(A') \neq C(A)$, and if $Ae^k \in C(A')$ then $Ae^k \in Col(A', I^i)$, implying $Col(A', I^i) = Col(A, I^i)$, with $i \in \{1, 2, \ldots, m\}$ arbitrary; thus $C(A')$ coincides with $C(A)$ □

3. Extreme directions of $C(A)$

Extreme directions of $C(A)$ can be found among the column vectors of the matrix A. The next theorem formulates a sufficient condition for a column vector to be extreme in $C(A)$. It is not a necessary condition as will be shown by an example.

Theorem 5. *A column vector Ae^k is extreme in $C(A)$ if there is a constraint i such that e^k is the only element in the corresponding facet, i.e. $\{e^k\} = \{x \in \mathbb{R}^n_+ : Ax \leq Ae^k, (Ax)_i = (Ae^k)_i\}$.*

Proof. Suppose $Ae^k = b^1 + b^2$, with $b^1, b^2 \in C(A)$. Let z^r be an element of $\{x \in \mathbb{R}^n_+ : Ax \leq b^r$ and $(Ax)_k = b^r_k\}$, $r = 1, 2$. Then $A(z^1 + z^2) \leq b^1 + b^2 = Ae^k$ and $(A(z^1 + z^2))_i = b^1_i + b^2_i = (Ae^k)_i$. Hence, $z^1 + z^2 \in \{x \in \mathbb{R}^n_+ : Ax \leq Ae^k$ and $(Ax)_i = Ae^k_i\} = \{e^k\}$. Thus, using the fact that z^1 and z^2 are non-negative vectors we conclude that z^1 and z^2 are non-negative multiples of e^k. Furthermore, from $b^1 + b^2 = Ae^k = A(z^1 + z^2) = Az^1 + Az^2 \leq b^1 + b^2$ we conclude that Az^r equals b^r, $r = 1, 2$. Hence, b^1 and b^2 are non-negative multiples of Ae^k. This proves that Ae^k cannot be written as a non-trivial non-negative combination of elements of $C(A)$. The theorem now follows. □

For a row $i \in \{1, 2, \dots, m\}$ of A with positive coefficients the set $\{x \in \mathbb{R}^n_+ : (Ax)_i \le 1\}$ is the convex hull of the origin 0 and the elements $(a_{ik})^{-1}e^k$, $1 \le k \le n$. Thus, there is a sharp constraint vector, denoted by β^i, such that $\{x \in \mathbb{R}^n_+ : Ax \le \beta^i\} = \{x \in \mathbb{R}^n_+ : (Ax)_i \le 1\}$. The coefficients of this constraint vector are given by

$$\beta^i_j = \min\Big(\Big\{\beta \in \mathbb{R}_+ : \ (e^j)^t A \le \beta (e^i)^t A\Big\}\Big) \tag{2}$$

$$= \max\Big(0, \max_{1 \le k \le n} a_{jk}/a_{ik}\Big), \quad 1 \le j \le m. \tag{3}$$

Note that β^i_i equals 1.

Let $I(A)$ denote the set of row indices $i \in \{1, 2, \dots, m\}$ with $a_{ik} > 0$ for all $k \in \{1, 2, \dots, n\}$.

Theorem 6. *β^i is an extreme direction of $C(A)$ for all $i \in I(A)$.*

Proof. Let b be a sharp constraint vector and let $x^1, \dots, x^m$ be non-negative n-vectors such that $Ax^j \le b$ and $(Ax^j)_j = b_j$, $1 \le j \le m$. Using (2) we have $b_j = (e^j)^t A x^j \le (\beta^i_j (e^i)^t A) x^j = \beta^i_j (Ax)_i \le \beta^i_j b_i$, $1 \le j \le m$, and we conclude that for each $b \in C(A)$ it holds that

$$b \le b_i \beta^i \quad \text{for each } i \in I(A). \tag{4}$$

Now let $\beta^i = b^1 + b^2$ for two sharp constraint vectors b^1 and b^2. Applying (4) we obtain $\beta^i = b^1 + b^2 \le b^1_i \beta^i + b^2_i \beta^i = (b^1 + b^2)_i \beta^i = \beta^i_i \beta^i = \beta^i$. Hence, $b^1 = b^1_i \beta^i$ and $b^2 = b^2_i \beta^i$. This proves the extremality of β^i in $C(A)$. □

Consider the (3×3)-matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}.$$

Using (3) one easily obtains $\beta^1 = (1, 2, 4)^t$, $\beta^2 = (1, 1, 2)^t$ and $\beta^3 = (1, 1, 1)^t$, which are all column vectors of A. Each facet corresponding to one of the constraints in the system $(Ax \le (1, 1, 2)^t, x \ge 0)$ trivially contains e^2. With little effort one can check that the facet corresponding to the first constraint also contains e^1, and $\frac{1}{2}e^3$ is an element of the facets corresponding to the other two constraints. The condition mentioned in Theorem 5 is, therefore, not satisfied although the column vector $(1, 1, 2)^t$ is extreme since it equals β^2.

The next example shows that the sharp constraint set may possess extreme directions which do not correspond to the column vectors or to the vectors β^i, $i \in I(A)$.

Let b be a sharp constraint vector of the (4×3)-matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix}.$$

For each non-negative 3-vector x with $Ax \le b$ we have

$$(Ax)_4 = 3x_1 + x_2 + 3x_3 \le 3x_1 + 2x_2 + 3x_3 = (Ax)_1 + (Ax)_2 \le b_1 + b_2.$$

Then $b_4 \le b_1 + b_2$ since b is sharp. Similarly, $3b_1 \ge b_3$ and $3b_2 \ge b_3$. Hence, $C(A)$ is contained in $\{b \in \mathbb{R}^4 : \ b_1 + b_2 - b_4 \ge 0, 3b_1 - b_3 \ge 0, 3b_2 - b_3 \ge 0\}$. The latter set is a polyhedral cone for which $\delta = (1, 1, 3, 2)^t$ is an extreme direction. One easily checks that δ is an element of $C(A)$ and, therefore, it has to be extreme in $C(A)$ also. Note that it is not a multiple of one of the column vectors or the vectors $\beta^1 = (1, 2, 3, 3)^t$, $\beta^2 = (2, 1, 3, 3)^t$, $\beta^3 = (1, 1, 1, 1\frac{1}{2})^t$ and $\beta^4 = (1, 1, 3, 1)^t$.

The extremality of a column k in $Col(A)$ is equivalent to the property that the intersection of all facets in $\{x \in \mathbb{R}^n_+ : Ax \le Ae^k\}$ equals $\{e^k\}$. The condition stated in Theorem 5 is stronger and the next example shows that it cannot be weakened to the mentioned property. Let A' be the matrix in the previous example enlarged with the column vector $Ae^2 + \delta$; i.e.,

$$A' = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 3 & 2 & 6 \\ 3 & 1 & 3 & 3 \end{pmatrix}.$$

By straightforward calculations one shows that the facets in $\{x \in \mathbb{R}^4_+ : A'x \le A'e^4\}$ are $Conv(\{e^4, 2e^2, e^3, (\frac{1}{3}, 1, \frac{1}{3}, 0)\})$, $Conv(\{e^4, 2e^2, e^1, (\frac{1}{3}, 1, \frac{1}{3}, 0)\})$, $Conv(\{e^4, 2e^2\})$ and $Conv(\{e^4, e^1, e^3, (\frac{1}{3}, 1, 4\frac{1}{3}, 0)\})$. The intersection of these sets equals $\{e^4\}$, showing the extremality of $A'e^4$ in $Col(A')$. Of course, $A'e^4 = A'e^2 + \delta$ is not extreme in $C(A')$.

It is till now still a problem to characterize all extreme directions of $C(A)$ in terms of properties of the corresponding feasible region or of the matrix A.

4. The case $C(A) = Col(A)$

Let us call a set $C \in \mathbb{R}^k$ *maximum-closed* if for each two elements x, y of C also its coordinate-wise maximum $\max(x, y) = (\max(x_i, y_i))_{i \in \{1,2,\dots,k\}}$ is contained in C.

Examples of maximum-closed sets are the cones $Col(A, I^i)$ with $1 \le i \le m$. This follows from the observation that for arbitrary elements $Ax + z$ and $Ay + z'$ of $Col(A, I^i)$ with $x, y \in \mathbb{R}^n_+$ and $z, z' \in \mathbb{R}^m_+$, $z_i = z'_i = 0$, we have $\max(Ax + z, Ay + z') = Ax + z''$, with z'' the non-negative m-vector defined by $z''_j = \max(0, (A(y - x))_j + z'_j - z_j)$, $j \in \{1, 2, \dots, m\}$. Assuming that $(Ax)_i \ge (Ay)_i$ holds we have $z''_i = 0$, implying $Ax + z'' \in Col(A, I^i)$.

Intersections of maximum-closed sets are, again, maximum-closed. Therefore, $C(A)$ is maximum-closed by Lemma 2.

Moreover, we have

Theorem 7. *The sharp constraint set $C(A)$ equals $Col(A)$ if and only if $Col(A)$ is maximum-closed.*

Proof. The maximum-closedness of $C(A)$ proves the 'only if'-part of the theorem. Thus, suppose $Col(A)$ is maximum-closed. Let b be an arbitrary sharp constraint vector. For each $i \in \{1, 2, \dots, m\}$ an element exists, say $x^i \in \mathbb{R}^n_+$, such that $Ax^i \le b$ and $(Ax^i)_i = b_i$. The maximum-closedness of $Col(A)$ implies the existence of an element $\hat{x} \in \mathbb{R}^n_+$ such that $A\hat{x}$ equals the maximum of the vectors Ax^i, $i \in \{1, 2, \dots, m\}$. It easily follows that $A\hat{x} = b$ which proves the inclusion $C(A) \subseteq Col(A)$. Equality follows since $C(A) \supseteq Col(A)$ holds in general. □

In the first example of the previous section $Col(A)$ is maximum-closed since the maximum of each two column vectors is again a column vector. In fact the column vectors are queued in increasing order. Other examples of matrices A with $Col(A)$ maximum-closed are the $(2 \times n)$-matrices with non-negative entries. This follows from the fact that each cone included in $\mathbb{R}^2_+$ is maximum-closed.

Corollary 8. *$C(A) = Col(A)$ for a non-negative $(2 \times n)$-matrix A.*

For a square matrix A we have the following

Theorem 9. *Let A be an invertible $(m \times m)$-matrix. Then $C(A) = Col(A)$ if and only if each row vector of A^{-1} has at most one negative coefficient.*

Proof. One easily proves that a closed halfspace $\{z \in \mathbb{R}^m\colon y^t z \geq 0\}$ is maximum-closed if the normal vector $y \in \mathbb{R}^m$ has at most one negative coefficient. Therefore, if each row vector of A^{-1} has at most one negative coefficient then $\cap_{1 \leq i \leq m}\{z\colon (e^i)^t A^{-1} z \geq 0\}$ is maximum-closed. The latter set obviously equals $\{z\colon A^{-1}z \geq 0\}$ which coincides with $\{Ax\colon x \in \mathbb{R}^m_+\} = Col(A)$. Thus, $Col(A)$ is maximum-closed yielding $C(A) = Col(A)$ by Theorem 7.

Now suppose there is a row of $A^{-1} = (a'_{ij})_{i,j \in \{1,2,\dots,m\}}$ with at least two negative coefficients, say $a'_{m,i} < 0$ and $a'_{m,j} < 0$, $i \neq j$. Let z denote the sum of the columns of A except column m, $z = A(1, 1,\dots, 1, 0)^t$. For the vector $y = a'_{m,j}A^{-1}e^i - a'_{m,i}A^{-1}e^j$ we have $y_m = a'_{m,j}a'_{m,i} - a'_{m,j}a'_{m,i} = 0$; thus there is an $\varepsilon > 0$ such that both $x^1 = (1, 1,\dots,1, 0)^t + \varepsilon y$ and $x^2 = (1, 1,\dots,1, 0)^t - \varepsilon y$ are non-negative vectors. Then Ax^1, $Ax^2 \in Col(A)$. Now Ax^1 equals $z + \varepsilon Ay = z + \varepsilon a'_{m,j}e^i - \varepsilon a'_{m,i}e^j$ and $Ax^2 = z - \varepsilon a'_{m,j}e^i + \varepsilon a'_{m,i}e^j$. Therefore, the maximum of these two vectors equals $z' = z - \varepsilon a'_{m,j}e^i - \varepsilon a'_{m,i}e^j$. Note that $(A^{-1}z')_m = (A^{-1}z)_m - 2\varepsilon a'_{m,j}a'_{m,i} = 0 - 2\varepsilon a'_{m,j}a'_{m,i} < 0$ and, therefore, z' cannot be an element of $Col(A)$. This shows that $Col(A)$ is not maximum-closed, yielding $C(A) \neq Col(A)$. □

Theorem 9 can also be proven using Theorem 7 and the fact that a polyhedral cone is maximum-closed if and only if it can be written as a finite intersection of closed halfspaces with normal vectors for which there is at most one negative coefficient. The proof of the latter equivalence follows the same line as the second part of the proof of Theorem 9.

5. The case $C(A) = Cone(\{\beta^i\colon 1 \leq i \leq m\})$

In this section we characterize matrices A for which the sharp constraint set is generated by the vectors β^i, $i \in \{1, 2,\dots, m\}$. These vectors exist only if all coefficients of the matrix A are positive. We therefore assume now that A is positive. For matters of convenience we furthermore assume that A does not contain rows which are multiples of each other. Thus, $C(A)$ is fully dimensional according to Theorem 3.

Let Q denote the $(m \times m)$-matrix consisting of the columns $\beta^1, \beta^2,\dots, \beta^m$, i.e., $q_{ij} = \max_{1 \leq k \leq n}(a_{jk}/a_{ik})$ for $1 \leq i, j \leq m$. Of course, $C(A) \supseteq Col(Q)$.

Theorem 10. *The sharp constraint set of A equals the cone generated by the sharp constraint vectors $\beta^1, \beta^2,\dots, \beta^m$, i.e., $C(A) = Col(Q)$, if and only if the following two conditions are satisfied:*
(i) each column vector of A is contained in $Col(Q)$;
(ii) Q is invertible and each row vector of Q^{-1} contains at most one negative coefficient.

Proof. Let us first show the 'if'-part. Since each column vector of A is an element of $Col(Q)$ we have $C(Q) = C(A, Q)$ according to the second statement of Lemma 4. The column vectors of Q are elements of $C(A)$. Thus we also have $C(A) = C(A, Q)$ implying $C(A) = C(Q)$. Theorem 9 and (ii) imply $C(Q) = Col(Q)$. Hence, $C(A) = Col(Q)$.

The 'only if'-part. Suppose $C(A) = Col(Q)$. Now (i) is valid since $Col(A) \subseteq C(A) = Col(Q)$. Thus, we have

$$Col(Q) = C(A) = C(A, Q) = C(Q). \tag{5}$$

Now suppose $yQ = 0^t$ for an m-vector y. Then $y^t b = 0$ for each element b of $C(A)$. Using the full dimensionality of $C(A)$ we conclude that $y = 0$. Therefore, Q^{-1} exists and according to (5) and Theorem 9 it must fulfil the second condition in (ii). □

Little is known about necessary and sufficient conditions on the matrix A to assert that the corresponding matrix Q is invertible. Of course, it is necessary that no two row vectors of A are multiples of each other to assert the invertibility of Q. It can be shown that it is also sufficient in the case

A consists of maximally 4 rows. The following example illustrates that sufficiency does not hold in general.Consider the (5×5)-matrix

$$A=\begin{pmatrix}1&2&2&1&1\\2&1&2&1&1\\2&2&1&1&1\\2&2&2&1&2\\3&3&3&2&1\end{pmatrix}.$$

With little effort one shows that the matrix Q corresponding to A equals A itself. Note that $(1, 1, 1, -1, -1)A = 0^t$ which implies that Q is not invertible.

If A is a positive $(m \times 2)$-matrix for which none of the row vectors are multiples of each other, then A and the corresponding matrix Q fulfil conditions (i) and (ii) in theorem 10. To show this, we consider A to have the following normalized form:

$$A=\begin{pmatrix}1&1&\cdots&1\\a_1&a_2&\cdots&a_m\end{pmatrix}^t \quad \text{with } 0<a_1<a_2<\cdots<a_m.$$

Then $\beta_j^i = \max(1, a_j/a_i)$, $i,j \in \{1, 2,\dots, m\}$. Note that condition (i) is satisfied since the first column vector of A equals β^m and the second column vector of A equals $a_1\beta^1$. Now define scalars $c_1, c_2,\dots, c_m$ and $d_1, d_2,\dots, d_m$ as follows.

$$c_i=\begin{cases}0 & \text{if } i=1,\\ a_i/(a_i-a_{i-1}) & \text{if } 1<i\le m,\end{cases}$$

$$d_i=\begin{cases}a_i/(a_{i+1}-a_i) & \text{if } 1\le i<m,\\ -1 & \text{if } i=m.\end{cases}$$

Let

$$Q'=\begin{pmatrix}-c_1-d_1 & d_1 & 0 & \cdots & 0\\ c_2 & -c_2-d_2 & d_2 & \cdots & 0\\ 0 & c_3 & -c_3-d_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -c_m-d_m\end{pmatrix}.$$

With straightforward calculations one shows that $Q'Q$ equals the identity matrix, i.e., $Q' = Q^{-1}$. The only negative coefficients of Q' are the diagonal elements. Therefore, Q fulfils (ii) of Theorem 10.

Corollary 11. *$C(A) = Col(Q)$ for a positive $(m \times 2)$-matrix A.*

Let A be the (3×2)-matrix

$$\begin{pmatrix}1&\frac{1}{2}\\1&1\\1&3\end{pmatrix}.$$

Then Q equals

$$\begin{pmatrix}1&1&1\\2&1&1\\6&3&1\end{pmatrix}.$$

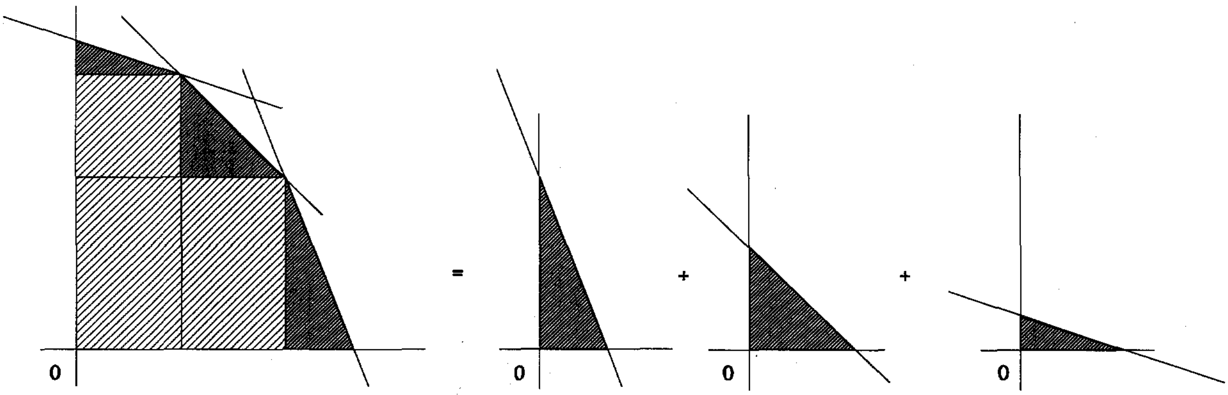

Fig. 1. Decomposition of a 2-dimensional feasible region.

From A we derive $c_1 = 0$, $c_2 = 2$, $c_3 = \frac{3}{2}$ and $d_1 = 1$, $d_2 = \frac{1}{2}$ and $d_3 = -1$. Then

$$Q' = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -\frac{5}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Applying Theorem 10 we conclude that $C(A) = \{b \in \mathbb{R}^3_+ : Q'b \geq 0\}$.

It is generally known that a 2-dimensional feasible region can be decomposed in a way as is illustrated in Fig. 1. If the feasible region in the figure equals $\{x \in \mathbb{R}^2_+ : Ax \leq b\}$ for a positive $(m \times 2)$-matrix A and sharp constraint vector b, then the triangular like sets on the right-hand side are the sets $(Q'b)_i\{x \in \mathbb{R}^2_+ : Ax \leq \beta^i\} = \{x \in \mathbb{R}^2_+ : (Ax)_i \leq (Q'b)_i\}$, $i \in \{1, 2, \ldots, m\}$.

6. An upper bound set for $C(A)$

In this section we still assume that A is a positive matrix.

For a constraint vector b for which (4) is not fulfilled, i.e., there is an index $j \in \{1, 2, \ldots, m\}$ with $b_j > b_i \beta^i_j$, the facet corresponding to constraint j in the system $(Ax \leq b,\ x \geq 0)$ is empty. This follows from the fact that for each n-vector $x \geq 0$ with $Ax \leq b$ we have $(e^j)^t Ax \leq (\beta^i_j (e^i)^t A)x \leq \beta^i_j b_i < b_j$.

Let $M(A)$ denote the set of all m-vectors b for which (4) holds, i.e.,

$$M(A) = \{b \in \mathbb{R}^m :\quad b \leq b_i \beta^i, \quad 1 \leq i \leq m\}.$$

The following lemma describes the elements of $M(A)$ more specifically.

Lemma 12. *$b \in M(A)$ if and only if each two of the sets $\{x \in \mathbb{R}^n_+ : (Ax)_i = b_i\}$, $1 \leq i \leq m$, have a non-empty intersection.*

Proof. First observe that the non-emptiness of the intersection of $\{x \in \mathbb{R}^n_+ : (Ax)_i = b_i\}$ and $\{x \in \mathbb{R}^n_+ : (Ax)_j = b_j\}$, with $i \neq j$ arbitrary, is equivalent to the existence of two indices k and k' with $(a_{ik})^{-1} b_i \leq (a_{jk})^{-1} b_j$ and $(a_{ik'})^{-1} b_i \geq (a_{jk'})^{-1} b_j$ (see Fig. 2). Using (3) we obtain the equivalence of $\{x \in \mathbb{R}^n_+ : (Ax)_i = b_i\} \cap \{x \in \mathbb{R}^n_+ : (Ax)_j = b_j\} \neq \emptyset$ with $b_i \leq b_j \beta^j_i$ and $b_j \leq b_i \beta^i_j$. From this equivalence the lemma easily follows. □

The sharp constraint set $C(A)$ is contained in $M(A)$ since we showed that each sharp constraint vector fulfils (4). The advantage of the set $M(A)$ is, in comparison with $C(A)$, the computational ease to

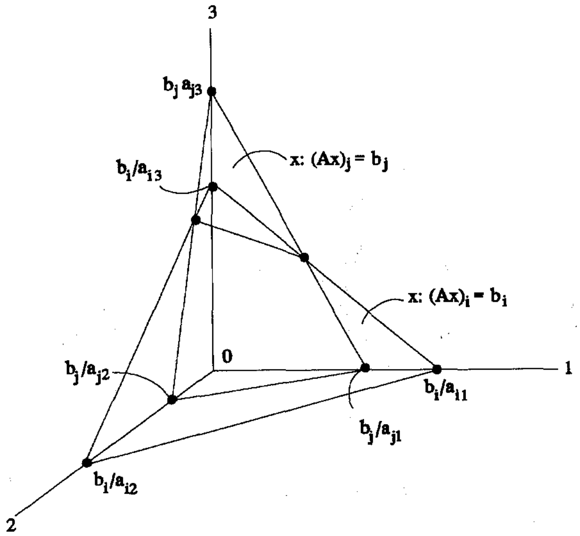

Fig. 2. Intersection of two hyperplanes in $\mathbb{R}^3_+$.

test whether a constraint vector is contained in $M(A)$. Therefore, one may be interested in the class of matrices A for which the sharp constraint set equals $M(A)$. A characterization will be presented.

First let us consider $M(A)$ more closely. $M(A)$ is a polyhedral cone and it is easily checked that the vectors β^i, $i \in I(A)$, are extreme directions. In fact $M(A)$ is the smallest cone containing these vectors and which is *minimum-closed*: if $b^1, b^2 \in M(A)$ then $\min(b^1, b^2) = (\min(b_i^1, b_i^2))_{i \in \{1,2,\ldots,m\}}$ is an element of $M(A)$. This follows easily from the fact that each minimum-closed cone M containing the vectors β^i, $1 \le i \le m$, also contains $\tilde{b} = \min_{1 \le i \le m} b_i \beta^i$, with $b \in \mathbb{R}^m_+$ arbitrary. Now $\tilde{b} = b$ if and only if $b \in M(A)$ and, thus, M contains $M(A)$ as a subset.

We conclude that the following theorem holds.

Theorem 13. *$C(A) = M(A)$ if and only if $C(A)$ is minimum-closed.*

Note that $M(A)$ equals $\{b \in \mathbb{R}^m: b \ge b_j \alpha^j, 1 \le j \le m\}$ with the m-vectors α^j, $1 \le j \le m$, defined as

$$\alpha_i^j = \left(\beta_j^i\right)^{-1}, \quad i \in \{1, 2, \ldots, m\}. \tag{6}$$

One easily shows that the vectors α^j, $1 \le j \le m$, are (extreme) elements of $M(A)$ and by the same reasoning as above one shows that $M(A)$ is the smallest maximum-closed cone containing these vectors.

With the help of these vectors α^j, $1 \le j \le m$, we formulate a method to decide whether or not $C(A)$ is minimum-closed.

Theorem 14. *$C(A)$ is minimum-closed if and only if each vector α^j, $1 \le j \le m$, is a sharp constraint vector of A.*

Proof. Let $C(A)$ be minimum-closed. Then $C(A) = M(A)$ implying $\alpha^j \in C(A)$, $1 \le j \le m$.

Now suppose $\alpha^j \in C(A)$, $1 \le j \le m$. Now $C(A)$ is maximum-closed and, therefore, it must contain as a subset the smallest maximum-closed cone containing α^j, $1 \le j \le m$; i.e., $M(A) \subseteq C(A)$. Thus, equality $M(A) = C(A)$ holds. □

The next lemma shows that one easily detects the sharpness of the vectors α^j, $1 \le j \le m$.

Lemma 15. *For* $j \in \{1, 2, \dots, m\}$ *the following three statements are equivalent:*
a) $\alpha^j = (a_{jk})^{-1}Ae^k$ *for a column* $k \in \{1, 2, \dots, n\}$,
b) $\alpha^j \in C(A)$,
c) *The facet corresponding to constraint* j *in* $\{x \in \mathbb{R}^n_+ \colon Ax \le \alpha^j\}$ *is non-epmty.*

Proof. Note that the only non-obvious part of the proof is the one of c) $\Rightarrow$ a). Thus, let x^j be a non-negative n-vector with $Ax^j \le \alpha^j$ and $(Ax^j)_j = \alpha^j_j = (\beta^j_j)^{-1} = 1$. According to (2) the n-vector $\beta^i_j(e^i)^tA - (e^j)^tA$ is non-negative for each $i \in \{1, 2, \dots, m\}$. Therefore,

$$0 \le \left(\beta^i_j(e^i)^tA - (e^j)^tA\right)^t x^j = \beta^i_j(Ax^j)_i - (Ax^j)_j \le \beta^i_j\alpha^j_i - \alpha^j_j = 0$$

for each $i \in \{1, 2, \dots, m\}$. We conclude that for a positive coefficient of x^j the corresponding coefficient of each of the vectors $\beta^i_j(e^i)^tA - (e^j)^tA$, $1 \le i \le m$, equals 0 implying that for a $k \in \{1, 2, \dots, n\}$ with $x^j_k > 0$ we must have $0 = (\beta^i_j(e^i)^tA - (e^j)^tA)_k = (\alpha^j_i)^{-1}a_{ik} - a_{jk}$, with $i \in \{1, 2, \dots, m\}$ arbitrary. Therefore, $\alpha^j = (a_{jk})^{-1}Ae^k$. □

Consider the (3×3)-matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}.$$

Then $\beta^1 = (1, 4, 3)^t$, $\beta^2 = (1, 1, 1\frac{1}{2})^t$ and $\beta^3 = (1, 2, 1)^t$. Thus, $\alpha^1 = (1, 1, 1)^t$ which equals the first column vector of A, $\alpha^2 = (\frac{1}{4}, 1, \frac{1}{2})^t = \frac{1}{4}(1, 4, 2)^t = \frac{1}{4}Ae^3$ and $\alpha^3 = (\frac{1}{3}, \frac{2}{3}, 1)^t = \frac{1}{3}Ae^2$. Hence, $C(A)$ must be minimum-closed implying $C(A) = M(A)$.

Using Lemma 12 one easily shows that whenever we have $C(A) = M(A)$, each facet of the feasible region corresponding to a sharp constraint vector contains an element of one of the axes of $\mathbb{R}^n_+$. This property trivially holds for positive $(2 \times n)$-matrices A. To show that in this case $C(A)$ equals $M(A)$ we assume that A admits the normalized form

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}, \quad \text{with } 0 < a_1 < a_2 < \cdots < a_n.$$

The 2-vectors β^1 and β^2 equal $(1, a_n)^t$ and $(a_1^{-1}, 1)^t$. Therefore, $\alpha^1 = (1, a_1)^t = Ae^1$ and $\alpha^2 = (a_n^{-1}, 1) = a_n^{-1}Ae^n$. We conclude that $C(A)$ is minimum-closed, yielding $C(A) = M(A)$.

7. Summary

We have shown that for an $(m \times n)$-matrix A the set of constraint vectors

$$C(A) = \{b \in \mathbb{R}^m \colon \ \{x \in \mathbb{R}^n_+ \colon \ Ax \le b \text{ and } (Ax)_i = b_i\} \ne \emptyset, \quad 1 \le i \le m\}$$

is a maximum-closed polyhedral cone, which is fully dimensional whenever no two rows are multiples of each other.

A column vector Ae^k of A is extreme in $C(A)$ if one of the facets of the corresponding feasible region consists only of e^k. For a row $i \in \{1, 2, \dots, m\}$ with only positive coefficients the constraint vector $\beta^i \in C(A)$, with $\{x \in \mathbb{R}^n_+ \colon Ax \le \beta^i\} = \{x \in \mathbb{R}^n_+ \colon (Ax)_i \le 1\}$, is an extreme direction of $C(A)$. It is an open problem to characterize all extreme directions in terms of the corresponding feasible regions or the matrix A.

We have shown that $C(A)$ equals the cone $Col(A)$ generated by the columns of A if and only if $Col(A)$ is maximum-closed.

Assuming that the matrix A consists of positive coefficients and without rows which are multiples of each other, the constraint vector set $C(A)$ equals the cone generated by the constraint vectors

$\beta^1, \beta^2, \ldots, \beta^m$, i.e., $C(A) = Col(Q)$ with $Q = (\beta^1, \ldots, \beta^m)$, if and only if $Col(A) \subseteq Col(Q)$ and Q^{-1} exists with at most one negative coefficient in each row. We have discussed the case $n = 2$ in which equality $C(A) = Col(Q)$ holds for each matrix A fulfilling our assumptions.

Again assuming the positiveness of A, each element b of $C(A)$ fulfils the inequalities $b \leq b_i \beta^i$, with $i \in \{1, 2, \ldots, m\}$. Therefore, the minimum-closed polyhedral cone

$$M(A) = \{b \in \mathbb{R}^m: \quad b \leq b_i \beta^i, \quad 1 \leq i \leq m\}$$

contains $C(A)$ as a subset. We have $C(A) = M(A)$ if and only if the vectors α^j, $j \in \{1, 2, \ldots, m\}$, defined by $\alpha^j_i = (\beta^i_j)^{-1}$ for $i \in \{1, 2, \ldots, m\}$, are multiples of some of the column vectors of A.

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